# PHILOSOPHICAL

# TRANSACTIONS

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*Meteorological Journal, kept at the Apartments of the Royal Society, by order of the President and Council.*

# PHILOSOPHICAL TRANSACTIONS.

## XV.—*Researches in Physical Astronomy.* By J. W. Lubbock, *Esq. V.P. and Treas. R.S.*

Read May 19, 1831.

### *On the Theory of the Moon.*

THE method pursued by Clairaut in the solution of this important problem of Physical Astronomy, consists in the integration of the differential equations furnished by the principles of dynamics, upon the hypothesis that in the gravitation of the celestial bodies the force varies inversely as the square of the distance, and in which the true longitude of the moon is the independent variable; the time is thus obtained in terms of the true longitude, and by the reversion of series the longitude is afterwards obtained in terms of the time, which is necessary for the purpose of forming astronomical tables. But while on the one hand this method possesses the advantage, that the disturbing function can be developed with somewhat greater facility in terms of the true longitude of the moon than in terms of the mean longitude, yet on the other hand, the differential equations in which the true longitude is the independent variable are far more complicated than those in which the time is the independent variable. The latter equations are used in the planetary theory; so that the method of Clairaut has the additional inconvenience, that while the lunar theory is a particular case of the problem of the three bodies, one system of equations is used in this case, and another in the case of the planets.

The method of Clairaut has been adopted, however, by Mayer, by Laplace, and by M. Damoiseau. The last-mentioned author has arranged his results with remarkable clearness, so that any part of his processes may be easily verified by any one who does not shrink from this gigantic undertaking; and the immense labour which this method requires, when all sensible quantities

are retained, may be seen in his invaluable memoir. Mr. Brice Bronwin has recently communicated to the Society a lunar theory, in which the same method is adopted.

Having reflected much upon the difficulties of this problem, I am led to believe that the integration of the differential equations in which the time is the independent variable, is at least as easy as the method hitherto, I think, solely employed, and I now have the honour to submit to the Society a lunar theory founded upon this integration, which is in fact merely an extension of the equations given in my Researches in Physical Astronomy, already printed, by embracing those terms which, in consequence of the magnitude of the eccentricity of the moon's orbit, are sensible; and the suppression of those, on the other hand, which are insensible on account of the great distance of the sun, the disturbing body. By means of the Table which I have given (Table II.), the developments may all be effected at once with the greatest facility.

The first column contains the indices, which I have employed to distinguish the inequalities. The numbers in the second column are the indices affixed by M. Damoiseau, in the Mém. sur la Théor. de la Lune, p. 547. to the inequalities of longitude.

$$t^* = nt - n_1 t, \quad x = cnt - \varpi, \quad z = n_1 t - \varpi_1, \quad y = gnt - \nu.$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | .. | 0 | 21 | 45 | $2t-3x$ | 42 | 73 | $2t-3x-z$ |
| 1 | 30 | $2t$† | 22 | 46 | $2t+3x$ | 43 | .. | $2t+3x+z$ |
| 2 | 1 | $x$ | 23 | 21 | $2x+z$ | 44 | 26 | $3x-z$ |
| 3 | 31 | $2t-x$‡ | 24 | 53 | $2t-2x-z$ | 45 | .. | $2t-3x+z$ |
| 4 | 32 | $2t+x$ | 25 | 54 | $2t+2x+z$ | 46 | .. | $2t+3x-z$ |
| 5 | 16 | $z$ § | 26 | 20 | $2x-z$ | 47 | .. | $2x+2z$ |
| 6 | 33 | $2t-z$ | 27 | 51 | $2t-2x+z$ | 48 | 75 | $2t-2x-2z$ |
| 7 | 34 | $2t+z$ | 28 | 52 | $2t+2x-z$ | 49 | .. | $2t+2x+2z$ |
| 8 | 2 | $2x$ | 29 | 23 | $x+2z$ | 50 | .. | $2x-2z$ |
| 9 | 35 | $2t-2x$ | 30 | 59 | $2t-x-2z$ | 51 | .. | $2t-2x+2z$ |
| 10 | 36 | $2t+2x$ | 31 | .. | $2t+x+2z$ | 52 | .. | $2t+2x-2z$ |
| 11 | 19 | $x+z$ | 32 | 22 | $x-2z$ | 53 | .. | $x+3z$ |
| 12 | 41 | $2t-x-z$ | 33 | 61 | $2t-x+2z$ | 54 | .. | $2t-x-3z$ |
| 13 | 42 | $2t+x+z$ | 34 | 60 | $2t+x-2z$ | 55 | .. | $2t+x+3z$ |
| 14 | 18 | $x-z$ | 35 | .. | $3z$ | 56 | .. | $x-3z$ |
| 15 | 39 | $2t-x+z$ | 36 | .. | $2t-3z$ | 57 | .. | $2t-x+3z$ |
| 16 | 40 | $2t+x-z$ | 37 | .. | $2t+3z$ | 58 | .. | $2t+x-3z$ |
| 17 | 17 | $2z$ | 38 | 9 | $4x$ | 59 | .. | $4z$ |
| 18 | 43 | $2t-2z$ | 39 | 67 | $2t-4x$ | 60 | .. | $2t-4z$ |
| 19 | 44 | $2t+2z$ | 40 | .. | $2t+4x$ | 61 | .. | $2t+4z$ |
| 20 | 4 | $3x$ | 41 | 27 | $3x+z$ | 62 | 3 | $2y$ |

* Inconvenience arises from using the letter $t$ in this acceptation. I have done so in order to conform to the notation of M. Damoiseau. † Variation. ‡ Evection. § Annual Equation.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 63 | 37 | $2t-2y$ | 105 | 84 | $t+z$ | 146 | .... | $y$ |
| 64 | 38 | $2t+2y$ | 106 | 85 | $t-2x$ | 147 | .... | $2t-y$ |
| 65 | 5 | $x-2y$ | 107 | 86 | $t+2x$ | 148 | .... | $2t+y$ |
| 66 | 6 | $x+2y$ | 108 | 91 | $t-x-z$ | 149 | .... | $x-y$ |
| 67 | 49 | $2t-x-2y$ | 109 | 92 | $t+x+z$ | 150 | .... | $x+y$ |
| 68 | 47 | $2t-x+2y$ | 110 | 89 | $t-x+z$ | 151 | .... | $2t-x-y$ |
| 69 | 48 | $2t+x-2y$ | 111 | .... | $t+x-z$ | 152 | .... | $2t-x+y$ |
| 70 | 50 | $2t+x+2y$ | 112 | .... | $t-2z$ | 153 | .... | $2t+x-y$ |
| 71 | 24 | $z-2y$ | 113 | .... | $t+2z$ | 154 | .... | $2t+x+y$ |
| 72 | 25 | $z+2y$ | 114 | .... | $t-2y$ | 155 | .... | $z-y$ |
| 73 | 57 | $2t-z-2y$ | 115 | .... | $t+2y$ | 156 | .... | $z+y$ |
| 74 | 56 | $2t-z+2y$ | 116 | 100 | $3t$ | 157 | .... | $2t-z-y$ |
| 75 | 55 | $2t+z-2y$ | 117 | 101 | $3t-x$ | 158 | .... | $2t-z+y$ |
| 76 | 58 | $2t+z+2y$ | 118 | 102 | $3t+x$ | 159 | .... | $2t+z-y$ |
| 77 | 7 | $2x-2y$ | 119 | 103 | $3t-z$ | 160 | .... | $2t+z+y$ |
| 78 | 8 | $2x+2y$ | 120 | 104 | $3t+z$ | 161 | .... | $2x \quad y$ |
| 79 | 65 | $2t-2x-2y$ | 121 | .... | $3t-2x$ | 162 | .... | $2x+y$ |
| 80 | 63 | $2t-2x+2y$ | 122 | .... | $3t+2x$ | 163 | .... | $2t-2x-y$ |
| 81 | 64 | $2t+2x-2y$ | 123 | .... | $3t-x-z$ | 164 | .... | $2t-2x+y$ |
| 82 | .. | $2t+2x+2y$ | 124 | .... | $3t+x+z$ | 165 | .... | $2t+2x-y$ |
| 83 | .. | $x+z-2y$ | 125 | .... | $3t-x+z$ | 166 | .... | $2t+2x+y$ |
| 84 | .. | $x+z+2y$ | 126 | .... | $3t+x-z$ | 167 | .... | $x+z-y$ |
| 85 | .. | $2t-x-z-2y$ | 127 | .... | $3t-2z$ | 168 | .... | $x+z+y$ |
| 86 | .. | $2t-x-z+2y$ | 128 | .... | $3t+2z$ | 169 | .... | $2t-x-z-y$ |
| 87 | .. | $2t+x+z-2y$ | 129 | .... | $3t-2y$ | 170 | .... | $2t-x-z+y$ |
| 88 | .. | $2t+x+z+2y$ | 130 | .... | $3t+2y$ | 171 | .... | $2t+x+z-y$ |
| 89 | .. | $x-z-2y$ | 131 | 120 | $4t$ | 172 | .... | $2t+x+z+y$ |
| 90 | .. | $x-z+2y$ | 132 | 121 | $4t-x$ | 173 | .... | $x-z-y$ |
| 91 | .. | $2t-x+z-2y$ | 133 | 122 | $4t+x$ | 174 | .... | $x-z+y$ |
| 92 | .. | $2t-x+z+2y$ | 134 | 123 | $4t-z$ | 175 | .... | $2t-x+z-y$ |
| 93 | .. | $2t+x-z-2y$ | 135 | 124 | $4t+z$ | 176 | .... | $2t-x+z+y$ |
| 94 | .. | $2t+x-z+2y$ | 136 | 125 | $4t-2x$ | 177 | .... | $2t+x-z-y$ |
| 95 | .. | $2z-2y$ | 137 | 126 | $4t+2x$ | 178 | .... | $2t+x-z+y$ |
| 96 | .. | $2z+2y$ | 138 | 131 | $4t-x-z$ | 179 | .... | $2z-y$ |
| 97 | .. | $2t-2z-2y$ | 139 | .... | $4t+x+z$ | 180 | .... | $2z+y$ |
| 98 | .. | $2t-2z+2y$ | 140 | 129 | $4t-x+z$ | 181 | .... | $2t-2z-y$ |
| 99 | .. | $2t+2z-2y$ | 141 | .... | $4t+x-z$ | 182 | .... | $2t-2z+y$ |
| 100 | .. | $2t+2z+2y$ | 142 | .... | $4t-2z$ | 183 | .... | $2t+2z-y$ |
| 101 | 80 | $t$* | 143 | .... | $4t+2z$ | 184 | .... | $2t+2z+y$ |
| 102 | 81 | $t-x$ | 144 | 127 | $4t-2y$ | 185 | .... | $t-y$ |
| 103 | 82 | $t+x$ | 145 | .... | $4t+2y$ | 186 | .... | $t+y$ |
| 104 | 83 | $t-z$ | | | | | | |

$$\cos 2t \cos 2t = \frac{1}{2} \cos 4t + \frac{1}{2}$$
$$[131] \qquad [0]$$
$$\cos 2t \cos x = \frac{1}{2} \cos (2t + x) + \frac{1}{2} \cos (-2t + x)$$
$$[4] \qquad [-3]$$

Hence the multiplication of $\cos 2t$ by $\cos 2t$ produces the arguments 131 and 0, similarly the multiplication of $\cos x$ by $\cos 2t$ produces the arguments 4 and $-3$; proceeding in this way the following Table was formed, by writing down the indices instead of the arguments themselves.

* Parallactic inequality.

## Table I.

Showing the arguments which result from the combination of the arguments 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17, 20, 35, 62, 101, 146 and 147, with the arguments 1, 2, 3, &c. by addition and subtraction.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 14 | 17 | 20 | 35 | 62 | 101 | 146 | 147 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 131 | 4 | 133 | 133 | 7 | 134 | 135 | 10 | 136 | 137 | 13 | 138 | 16 | 19 | 22 | 37 | 64 | 116 | 148 | ..... | 1 |
| | 0 | 3 | 2 | 2 | 6 | 5 | 5 | 9 | 8 | 8 | 12 | 11 | 15 | 18 | 21 | 36 | 63 | 101 | 147 | 146 | |
| 2 | 4 | 8 | 1 | 10 | 11 | 16 | 13 | 20 | 3 | 22 | 23 | 6 | 26 | 29 | 38 | 53 | 66 | 103 | 150 | 153 | 2 |
| | 3 | 0 | 9 | 1 | 14 | 12 | 15 | 2 | 21 | 4 | 5 | 24 | 5 | 32 | 8 | 56 | 65 | 102 | 149 | 151 | |
| 3 | 132 | 1 | 136 | 131 | 15 | 138 | 140 | 4 | ..... | 133 | 7 | ..... | 6 | 33 | 10 | 57 | 68 | 117 | 152 | ..... | 3 |
| | 2 | 9 | 0 | 8 | 12 | 14 | 11 | 21 | 2 | 20 | 24 | 5 | 27 | 30 | 39 | 54 | 67 | 102 | 151 | 149 | |
| 4 | 133 | 10 | 131 | 137 | 13 | 141 | 139 | 22 | 132 | ..... | 25 | 134 | 28 | 31 | 40 | 55 | 70 | 118 | 154 | ..... | 4 |
| | 2 | 1 | 8 | 0 | 16 | 11 | 14 | 3 | 20 | 2 | 6 | 23 | 7 | 34 | 9 | 58 | 69 | 103 | 153 | 150 | |
| 5 | 7 | 11 | 15 | 13 | 17 | 1 | 19 | 23 | 27 | 25 | 29 | 3 | 2 | 35 | 41 | 59 | 72 | 105 | 156 | 159 | 5 |
| | 6 | 14 | 12 | 16 | 0 | 18 | 1 | 26 | 24 | 28 | 2 | 30 | 32 | 5 | 44 | 17 | 71 | 104 | 155 | 157 | |
| 6 | 134 | 16 | 138 | 141 | 1 | 142 | 131 | 28 | ..... | ..... | 4 | ..... | 34 | 7 | 46 | 19 | 74 | 119 | 158 | ..... | 6 |
| | 5 | 12 | 14 | 11 | 18 | 0 | 17 | 24 | 26 | 22 | 30 | 2 | 3 | 36 | 42 | 60 | 73 | 104 | 157 | 155 | |
| 7 | 135 | 13 | 140 | 139 | 19 | 131 | 143 | 25 | ..... | ..... | 31 | 132 | 4 | 37 | 43 | 61 | 76 | 120 | 160 | ..... | 7 |
| | 5 | 15 | 11 | 14 | 1 | 17 | 0 | 27 | 23 | 26 | 3 | 29 | 33 | 6 | 45 | 18 | 75 | 105 | 159 | 156 | |
| 8 | 10 | 20 | 4 | 22 | 23 | 28 | 25 | 38 | 1 | ..... | 41 | 16 | 44 | 47 | ..... | ..... | 78 | 107 | 162 | 165 | 8 |
| | 9 | 2 | 21 | 3 | 26 | 24 | 27 | 0 | 39 | 1 | 14 | 42 | 11 | 50 | ..... | ..... | 77 | 106 | 161 | 163 | |
| 9 | 136 | 3 | ..... | 132 | 27 | ..... | ..... | 1 | ..... | 131 | 15 | ..... | 12 | 51 | ..... | ..... | 80 | 121 | 164 | ..... | 9 |
| | 8 | 21 | 2 | 20 | 24 | 26 | 23 | 39 | 0 | 38 | 42 | 14 | 45 | 48 | ..... | ..... | 79 | 106 | 163 | 161 | |
| 10 | 137 | 22 | 133 | ..... | 25 | ..... | ..... | 40 | 131 | ..... | 43 | 141 | 46 | 49 | ..... | ..... | 82 | 122 | 166 | ..... | 10 |
| | 8 | 4 | 20 | 2 | 28 | 23 | 26 | 1 | 38 | 0 | 16 | 41 | 13 | 52 | ..... | ..... | 81 | 107 | 165 | 162 | |
| 11 | 13 | 23 | 7 | 25 | 29 | 4 | 31 | 41 | 15 | 43 | 47 | 1 | 8 | 53 | ..... | ..... | 84 | 109 | 168 | 171 | 11 |
| | 12 | 5 | 24 | 6 | 2 | 30 | 3 | 14 | 42 | 16 | 0 | 48 | 17 | 14 | ..... | ..... | 83 | 108 | 167 | 169 | |
| 12 | 138 | 6 | ..... | 134 | 3 | ..... | 132 | 16 | ..... | 141 | 1 | ..... | 18 | 15 | ..... | ..... | 86 | 123 | 170 | ..... | 12 |
| | 11 | 24 | 5 | 23 | 30 | 2 | 29 | 42 | 14 | 41 | 48 | 0 | 9 | 54 | ..... | ..... | 85 | 108 | 169 | 167 | |
| 13 | 139 | 25 | 135 | ..... | 31 | 133 | ..... | 43 | 140 | ..... | 49 | 131 | 10 | 55 | ..... | ..... | 88 | 124 | 172 | ..... | 13 |
| | 11 | 7 | 23 | 5 | 4 | 29 | 2 | 15 | 41 | 14 | 1 | 47 | 19 | 16 | ..... | ..... | 87 | 109 | 171 | 168 | |
| 14 | 16 | 26 | 6 | 28 | 2 | 34 | 4 | 44 | 12 | 46 | 8 | 18 | 50 | 11 | ..... | ..... | 90 | 111 | 174 | 177 | 14 |
| | 15 | 5 | 27 | 7 | 32 | 3 | 33 | 11 | 45 | 13 | 17 | 9 | 0 | 56 | ..... | ..... | 89 | 110 | 173 | 175 | |
| 15 | 140 | 7 | ..... | 135 | 33 | 132 | ..... | 13 | ..... | 139 | 19 | 136 | 1 | 57 | ..... | ..... | 92 | 125 | 176 | ..... | 15 |
| | 14 | 27 | 5 | 26 | 3 | 32 | 2 | 45 | 11 | 44 | 9 | 17 | 51 | 12 | ..... | ..... | 91 | 110 | 175 | 173 | |
| 16 | 141 | 28 | 134 | ..... | 4 | ..... | 133 | 46 | 138 | ..... | 10 | 142 | 52 | 13 | ..... | ..... | 94 | 126 | 178 | ..... | 16 |
| | 14 | 6 | 26 | 5 | 34 | 2 | 32 | 12 | 44 | 11 | 18 | 8 | 1 | 58 | ..... | ..... | 93 | 111 | 177 | 174 | |
| 17 | 19 | 29 | 33 | 31 | 35 | 7 | 37 | 47 | 51 | 49 | 53 | 15 | 11 | 59 | ..... | ..... | 96 | 113 | 180 | 183 | 17 |
| | 18 | 32 | 30 | 34 | 5 | 36 | 6 | 50 | 48 | 52 | 14 | 54 | 56 | 0 | ..... | ..... | 95 | 112 | 179 | 181 | |
| 18 | 142 | 34 | ..... | ..... | 6 | ..... | 134 | 52 | ..... | ..... | 16 | ..... | 58 | 1 | ..... | ..... | 98 | 127 | 182 | ..... | 18 |
| | 17 | 30 | 32 | 29 | 36 | 5 | 35 | 48 | 50 | 47 | 54 | 14 | 12 | 60 | ..... | ..... | 97 | 112 | 181 | 179 | |
| 19 | 143 | 31 | ..... | ..... | 37 | 135 | ..... | 49 | ..... | ..... | 55 | 140 | 13 | 61 | ..... | ..... | 100 | 128 | 184 | ..... | 19 |
| | 17 | 33 | 29 | 32 | 7 | 35 | 5 | 51 | 47 | 50 | 15 | 53 | 57 | 1 | ..... | ..... | 99 | 113 | 183 | 180 | |
| 20 | 22 | 38 | 10 | 40 | 41 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 20 |
| | 21 | 8 | 39 | 9 | 44 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | |
| 21 | ..... | 9 | ..... | 136 | 45 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 21 |
| | 20 | 39 | 8 | 38 | 42 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | |

TABLE I. (Continued.)

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 14 | 17 | 20 | 35 | 62 | 101 | 146 | 147 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 22 | ..... | 40 | ..... | ..... | 43 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 22 |
| | 20 | 10 | 38 | 8 | 46 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 23 | 25 | 41 | 13 | 43 | 47 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 23 |
| | 24 | 11 | 42 | 12 | 8 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 24 | ..... | 12 | ..... | 138 | 9 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 24 |
| | 23 | 42 | 11 | 41 | 48 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 25 | ..... | 43 | 139 | ..... | 49 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 25 |
| | 23 | 13 | 41 | 11 | 10 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 26 | 28 | 44 | 16 | 46 | 8 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 26 |
| | 27 | 14 | 45 | 15 | 50 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 27 | ..... | 15 | ..... | 140 | 51 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 27 |
| | 26 | 45 | 14 | 44 | 9 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 28 | ..... | 46 | 141 | ..... | 10 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 28 |
| | 26 | 16 | 44 | 14 | 52 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 29 | 31 | 47 | 19 | 49 | 53 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 29 |
| | 30 | 17 | 48 | 18 | 11 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 30 | ..... | 18 | ..... | 142 | 12 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 30 |
| | 29 | 48 | 17 | 47 | 54 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 31 | ..... | 49 | 143 | ..... | 55 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 31 |
| | 29 | 19 | 47 | 17 | 13 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 32 | 34 | 50 | 18 | 52 | 14 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 32 |
| | 33 | 17 | 51 | 19 | 56 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 33 | ..... | 19 | ..... | 143 | 57 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 33 |
| | 32 | 51 | 17 | 50 | 15 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 34 | ..... | 52 | 142 | ..... | 16 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 34 |
| | 32 | 18 | 50 | 17 | 58 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 35 | 37 | 53 | 57 | 58 | 59 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 35 |
| | 36 | 56 | 54 | 58 | 17 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 36 | ..... | 58 | ..... | ..... | 18 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 36 |
| | 35 | 54 | 56 | 53 | 60 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 37 | ..... | 55 | ..... | ..... | 61 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 37 |
| | 35 | 57 | 53 | 56 | 19 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 62 | 64 | 66 | 68 | 70 | 72 | 74 | 76 | 78 | 80 | 82 | 84 | 86 | 90 | 96 | ..... | ..... | ... | 115 | ... | 148 | 62 |
| | 63 | 65 | 67 | 69 | 71 | 73 | 75 | 77 | 79 | 81 | 83 | 85 | 89 | 95 | ..... | ..... | 0 | 114 | 146 | ..... | |
| 63 | 144 | 69 | ..... | ..... | 75 | ..... | ..... | 81 | ..... | ..... | 87 | ..... | 93 | 99 | ..... | ..... | 1 | 129 | 147 | ..... | 63 |
| | 62 | 67 | 65 | 66 | 73 | 71 | 72 | 79 | 77 | 78 | 85 | 83 | 91 | 97 | ..... | ..... | ... | 114 | ... | 146 | |
| 64 | 145 | 70 | ..... | ..... | 76 | ..... | ..... | 82 | ..... | ..... | 88 | ..... | 94 | 100 | ..... | ..... | ... | 130 | ... | ..... | 64 |
| | 62 | 68 | 66 | 65 | 74 | 72 | 71 | 80 | 78 | 77 | 86 | 84 | 92 | 98 | ..... | ..... | 1 | 115 | 148 | ..... | |
| 65 | 69 | 77 | 63 | 81 | 83 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 65 |
| | 68 | 62 | 77 | 64 | 89 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 66 | 70 | 78 | 64 | 82 | 84 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 66 |
| | 67 | 62 | 79 | 63 | 90 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 67 | ..... | 63 | ..... | 144 | 91 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 67 |
| | 66 | 79 | 62 | 78 | 85 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 68 | ..... | 64 | ..... | 145 | 92 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 68 |
| | 65 | 80 | 62 | 77 | 86 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 69 | ..... | 81 | 144 | ..... | 87 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 69 |
| | 65 | 63 | 77 | 62 | 93 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |

TABLE I. (Continued.)

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 14 | 17 | 20 | 35 | 62 | 101 | 146 | 147 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 70 | ..... | 82 | 145 | ..... | 88 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 70 |
| | 66 | 64 | 78 | 62 | 94 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 71 | 75 | 83 | 91 | 87 | 93 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 71 |
| | − 74 | − 90 | − 86 | − 94 | − 62 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 72 | 76 | 84 | 92 | 88 | 96 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 72 |
| | − 73 | − 89 | − 85 | − 93 | 62 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 73 | ..... | 93 | ..... | ..... | 63 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 73 |
| | − 72 | 85 | 89 | − 84 | 97 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 74 | ..... | 94 | ..... | ..... | 64 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 74 |
| | − 71 | 86 | 90 | − 83 | 98 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 75 | ..... | 87 | ..... | ..... | 99 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 75 |
| | 71 | 91 | 85 | − 90 | 63 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 76 | ..... | 88 | ..... | ..... | 100 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 76 |
| | 72 | 92 | 84 | − 89 | 64 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 101 | 116 | 103 | 117 | 118 | 105 | 119 | 120 | 107 | 121 | 122 | 109 | 123 | 111 | 113 | ... | ... | 115 | 1 | 186 | ..... | 101 |
| | − 101 | 102 | − 102 | − 103 | 104 | − 104 | − 105 | 106 | − 106 | − 107 | 108 | − 108 | 110 | 112 | ... | ... | 114 | 0 | 185 | − 185 | |
| 102 | 117 | 101 | 121 | 116 | 110 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 102 |
| | − 103 | 106 | − 101 | − 107 | 108 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 103 | 118 | 107 | 116 | 122 | 109 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 103 |
| | − 102 | 101 | − 106 | − 101 | 111 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 104 | 119 | 111 | 123 | 126 | 101 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 104 |
| | − 105 | 108 | − 110 | − 109 | 112 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 105 | 120 | 109 | 125 | 124 | 113 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 105 |
| | − 104 | 110 | − 108 | − 111 | 101 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 116 | ..... | 118 | ..... | ..... | 120 | ..... | ..... | 122 | ..... | ..... | 124 | ..... | 126 | 128 | ... | ... | 130 | 131 | ... | ..... | 116 |
| | 101 | 117 | 103 | 102 | 119 | 105 | 104 | 121 | 107 | 106 | 123 | 109 | 125 | 127 | ... | ... | 129 | 1 | ... | 186 | |
| 117 | ..... | 116 | ..... | ..... | 125 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 117 |
| | 102 | 121 | 101 | 106 | 123 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 118 | ..... | 122 | ..... | ..... | 124 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 118 |
| | 103 | 116 | 107 | 101 | 126 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 119 | ..... | 126 | ..... | ..... | 116 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 119 |
| | 104 | 123 | 111 | 108 | 127 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 120 | ..... | 124 | ..... | ..... | 128 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 120 |
| | 105 | 125 | 109 | 110 | 116 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 131 | ..... | 133 | ..... | ..... | 135 | ..... | ..... | 137 | ..... | ..... | 139 | ..... | 141 | 143 | ... | ... | 145 | ..... | ... | ..... | 131 |
| | 1 | 132 | 4 | 3 | 134 | 7 | 6 | 136 | 10 | 9 | 138 | 13 | 140 | 142 | ... | ... | 144 | 116 | ... | 148 | |
| 132 | ..... | 131 | ..... | ..... | 140 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 132 |
| | 3 | 136 | 1 | 9 | 138 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 133 | ..... | 137 | ..... | ..... | 139 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 133 |
| | 4 | 131 | 10 | 1 | 141 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 134 | ..... | 141 | ..... | ..... | 131 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 134 |
| | 6 | 138 | 16 | 12 | 142 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 135 | ..... | 139 | ..... | ..... | 143 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 135 |
| | 7 | 140 | 13 | 15 | 131 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 146 | 148 | 150 | 152 | 154 | 156 | 158 | 160 | 162 | 164 | 166 | 168 | 170 | 174 | 180 | ... | ... | ..... | 186 | 62 | 1 | 146 |
| | − 147 | − 149 | − 151 | − 153 | − 155 | − 157 | − 159 | − 161 | 163 | − 165 | − 167 | − 169 | − 173 | − 179 | ... | ... | − 146 | − 185 | 0 | − 63 | |
| 147 | ..... | 153 | ..... | ..... | 159 | ..... | ..... | 164 | ..... | ..... | 171 | ..... | 177 | 183 | ... | ... | 148 | ..... | 1 | 144 | 147 |
| | − 146 | 151 | 149 | − 150 | 157 | 155 | − 156 | 163 | 161 | − 162 | 169 | 167 | 175 | 181 | ... | ... | ..... | 185 | 63 | 0 | |

## TABLE I. (Continued.)

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 14½ | 17 | 20 | 35 | 62 | 101 | 146 | 147 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 148 | ..... | 154 | ..... | ..... | 160 | ..... | ..... | 166 | ..... | ..... | 172 | ..... | 178 | 184 | ... | ... | ..... | ..... | 64 | 131 | 148 |
| | 146 | 152 | 150 | −149 | 158 | 156 | −155 | 164 | 162 | −161 | 170 | 168 | 176 | 182 | ... | ... | 147 | 186 | 1 | 62 | |
| 149 | 153 | 161 | 147 | 165 | 167 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 149 |
| | −152 | −146 | −164 | −148 | 173 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 150 | 154 | 162 | 148 | 166 | 168 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 150 |
| | −151 | 146 | −163 | −147 | 174 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 151 | ..... | 147 | ..... | ..... | 175 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 151 |
| | −150 | 163 | −146 | −162 | 169 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 152 | ..... | 148 | ..... | ..... | 176 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 152 |
| | −149 | 164 | 148 | −161 | 170 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 153 | ..... | 165 | ..... | ..... | 171 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 153 |
| | 149 | 147 | 161 | −146 | 177 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 154 | ..... | 166 | ..... | ..... | 172 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 154 |
| | 150 | 148 | 162 | 146 | 178 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 155 | 159 | 167 | 175 | ..... | 179 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 155 |
| | −158 | −174 | −170 | −178 | −146 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 156 | 160 | 168 | 176 | 172 | 180 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 156 |
| | −157 | −173 | −169 | −177 | 146 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 157 | ..... | 177 | ..... | ..... | 147 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 157 |
| | −156 | 169 | 173 | −168 | 181 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 158 | ..... | 178 | ..... | ..... | 148 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 158 |
| | −155 | 170 | 174 | −167 | 182 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 159 | ..... | 171 | ..... | ..... | 183 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 159 |
| | 155 | 175 | 167 | −174 | 147 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |
| 160 | ..... | 172 | ..... | ..... | 184 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | 160 |
| | 156 | 176 | 168 | −173 | 148 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ... | ..... | |

| | | 38 | 59 | |
|---|---|---|---|---|
| 1 | 40 | | 61 | 1 |
| | 39 | | 60 | |

## Table II.

Showing the arguments which, by their combination with the arguments 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17, 20, 35, 62, 101, 146, 147, produce the arguments, 12, 3, &c. in the left hand column. This Table is formed from the preceding, by making the numbers in the left hand column in that Table change places with the rest. A full stop is placed after the figure where it does not occupy the same *cell* as in the preceding Table.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 14 | 17 | 20 | 35 | 62 | 101 | 146 | 147 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 2 | 133. | 6 | 5 | ..... | 9 | 8 | ..... | 12 | 11 | 15 | 18 | ... | ... | 63 | 101 | 147 | 146 | 1 |
| | 131 | 4 | 132 | − 2 | 7 | ..... | − 5 | 10 | ... | − 8 | 13 | ..... | 16 | 19 | ... | ... | 64 | 116 | 148 | ... | |
| 2 | 4. | 0 | − 9. | 10. | 14 | 16. | − 15. | ..... | ... | ..... | ..... | ..... | 5 | ..... | ... | ... | ... | ... | ... | ... | 2 |
| | − 3 | 8 | 1 | − 1 | 11 | − 12 | 13 | − 2 | 3 | − 4 | − 5 | 6 | ... | ..... | ... | ... | ... | ... | ... | ... | |
| 3 | 132. | 9 | 0 | 131. | 12 | ..... | ..... | ..... | 2 | ..... | ..... | 5 | ... | ..... | ... | ... | ... | ... | ... | ... | 3 |
| | − 2 | 1 | ..... | − 8 | 15 | − 14 | − 11 | 4 | ... | ..... | 7 | ..... | 6 | ..... | ... | ... | ... | ... | ... | ... | |
| 4 | 2 | 1 | 8 | 0 | 16 | 11 | 14 | 3 | ... | ..... | 6 | ..... | 7 | ..... | ... | ... | ... | ... | ... | ... | 4 |
| | 133 | 10 | 131 | ..... | 13 | ..... | ..... | ..... | ... | − 2 | ..... | ..... | ... | ..... | ... | ... | ... | ... | ... | ... | |
| 5 | 7. | − 14. | 15. | − 16. | 0 | − 18. | 19. | ..... | ... | ..... | ..... | ..... | ... | ..... | ... | ... | ... | ... | ... | ... | 5 |
| | − 6 | 11 | − 12 | 13 | 17 | 1 | − 1 | ..... | ... | ..... | − 2 | 3 | 2 | − 5 | ... | ... | ... | ... | ... | ... | |
| 6 | 134. | 12 | 14 | ..... | 18 | ..... | 131. | ..... | ... | ..... | ..... | 2 | 3 | ..... | ... | ... | ... | ... | ... | ... | 6 |
| | − 5 | 16 | ..... | − 11 | 1 | ..... | − 17 | ..... | ... | ..... | 4 | ..... | ... | 7 | ... | ... | ... | ... | ... | ... | |
| 7 | 5 | 15 | 11 | ..... | 1 | 17 | ..... | ..... | ... | ..... | 3 | ..... | ... | 6 | ... | ... | ... | ... | ... | ... | 7 |
| | 135 | 13 | ..... | − 14 | 19 | 131 | ..... | ..... | ... | ..... | ..... | ..... | 4 | ..... | ... | ... | ... | ... | ... | ... | |
| 8 | 10. | 2 | − 21. | 22. | 26 | ..... | ..... | ..... | ... | ..... | 14 | ..... | 11 | ..... | ... | ... | ... | ... | ... | ... | 8 |
| | − 9 | 20 | 4 | − 3 | 23 | ..... | ..... | ..... | 1 | − 1 | ..... | 16 | ... | ..... | − 2 | ... | ... | ... | ... | ... | |
| 9 | ..... | 21 | ..... | 132. | 24 | ..... | ..... | ..... | ... | ..... | ..... | ..... | ... | ..... | ... | ... | ... | ... | ... | ... | 9 |
| | − 8 | 3 | − 2 | − 20 | 27 | ..... | ..... | 1 | ... | 131 | 15 | −14 | 12 | ..... | 4 | ... | ... | ... | ... | ... | |
| 10 | 8 | 4 | 20 | 2 | 28 | ..... | ..... | 1 | ... | ..... | 16 | ..... | 13 | ..... | 3 | ... | ... | ... | ... | ... | 10 |
| | ..... | 22 | 133 | ..... | 25 | ..... | ..... | ..... | 131 | ..... | ..... | ..... | ... | ..... | ... | ... | ... | ... | ... | ... | |
| 11 | 13. | 5 | − 24. | 25. | 2 | ..... | ..... | ..... | ... | ..... | ..... | ..... | 17 | 14 | ... | ... | ... | ... | ... | ... | 11 |
| | − 12 | 23 | 7 | − 6 | 29 | 4 | − 3 | − 14 | 15 | − 16 | ..... | 1 | 8 | ..... | ... | ... | ... | ... | ... | ... | |
| 12 | ..... | 24 | ..... | 134. | 30 | ..... | ..... | ..... | 14 | ..... | ..... | ..... | 9 | ..... | ... | ... | ... | ... | ... | ... | 12 |
| | − 11 | 6 | − 5 | − 23 | 3 | − 2 | ..... | 16 | ... | ..... | 1 | ..... | 18 | 15 | ... | ... | ... | ... | ... | ... | |
| 13 | 11 | 7 | 23 | 5 | 4 | ..... | 2 | 15 | ... | ..... | 1 | ..... | 19 | 16 | ... | ... | ... | ... | ... | ... | 13 |
| | ..... | 25 | 135 | ..... | 31 | ..... | ..... | ..... | ... | − 14 | ..... | 131 | 10 | ..... | ... | ... | ... | ... | ... | ... | |
| 14 | 16. | 26. | − 27. | 28. | 32 | ..... | ..... | ..... | ... | ..... | − 17. | 18. | ... | ..... | ... | ... | ... | ... | ... | ... | 14 |
| | − 15 | − 5 | 6 | − 7 | 2 | − 3 | 4 | − 11 | 12 | − 13 | 8 | − 9 | ... | 11 | ... | ... | ... | ... | ... | ... | |
| 15 | ..... | 27 | 5 | 135. | 3 | ..... | ..... | ..... | 11 | ..... | 9 | 17 | ... | 12 | ... | ... | ... | ... | ... | ... | 15 |
| | − 14 | 7 | ..... | − 26 | 33 | ..... | − 2 | 13 | ... | ..... | 19 | ..... | 1 | ..... | ... | ... | ... | ... | ... | ... | |
| 16 | 14 | 6 | 26 | ..... | 34 | 2 | ..... | 12 | ... | ..... | 18 | 8 | 1 | ..... | ... | ... | ... | ... | ... | ... | 16 |
| | ..... | 28 | 134 | − 5 | 4 | ..... | ..... | ..... | ... | − 11 | 10 | ..... | ... | 13 | ... | ... | ... | ... | ... | ... | |
| 17 | 19. | − 32. | 33 | − 34. | 5 | ..... | ..... | ..... | ... | ..... | ..... | ..... | ... | 0 | ... | ... | ... | ... | ... | ... | 17 |
| | − 18 | 29 | − 30 | 31 | 35 | 7 | − 6 | ..... | ... | ..... | − 14 | 15 | 11 | ..... | ... | − 5 | ... | ... | ... | ... | |
| 18 | ..... | 30 | 32 | ..... | 36 | ..... | ..... | ..... | ... | ..... | ..... | 14 | 12 | ..... | ... | ... | ... | ... | ... | ... | 18 |
| | − 17 | 34 | ..... | − 29 | 6 | − 5 | ..... | ..... | ... | ..... | 16 | ..... | ... | 1 | ... | 7 | ... | ... | ... | ... | |
| 19 | 17 | 33 | 29 | ..... | 7 | ..... | 5 | ..... | ... | ..... | 15 | ..... | ... | 1 | ... | 6 | ... | ... | ... | ... | 19 |
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| 22 | 20 | 10 | | 8 | | | 4 | | 2 | | | | | 1 | | | | | | 22 |
| 23 | 25. | 11 | | | 8 | | 5 | | | 2 | | | | | | | | | | 23 |
| | − 24 | | 13 | − 12 | | 10 | − 9 | | 7 | − 6 | | 4 | | | | | | | | |
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| | − 23 | 12 | − 11 | | 9 | − 8 | | 6 | − 5 | | 3 | − 2 | | | | | | | | |
| 25 | 23 | 13 | | 11 | 10 | | 8 | 7 | | 5 | 4 | | | | | | | | | 25 |
| 26 | 28. | 14 | | | 8 | | | | | | | | 2 | | | | | | | 26 |
| | − 27 | | 16 | − 15 | | − 9 | 10 | − 5 | 6 | − 7 | | | | | | | | | | |
| 27 | | | | | 9 | | | | 5 | | | | | | | | | | | 27 |
| | − 26 | 15 | − 14 | | | | 8 | 7 | | | | | 3 | | | | | | | |
| 28 | 26 | 16 | | 14 | | 8 | | 6 | | | | | 4 | | | | | | | 28 |
| | | | | | 10 | | | | | − 5 | | | | | | | | | | |
| 29 | 31. | 17 | | | 11 | | | | | | 5 | | | 2 | | | | | | 29 |
| | − 30 | | 19 | − 18 | | 13 | − 12 | | | | | 7 | | | | | | | | |
| 30 | | | 18. | | | | | | | | | | | | | | | | | 30 |
| | − 29 | 18 | − 17 | | 12 | − 11 | | | | | 6 | − 5 | | 3 | | | | | | |
| 31 | 29 | 19 | | 17 | 13 | | 11 | | | | 7 | | | 4 | | | | | | 31 |
| 32 | 34. | | | | | | | | | | | | | | | | | | | 32 |
| | − 33 | − 17 | 18 | − 19 | 14 | − 15 | 16 | | | | | | − 5 | 2 | | | | | | |
| 33 | | | 17 | | 15 | | | | | | | | | 3 | | | | | | 33 |
| | − 32 | 19 | | | | | − 14 | | | | | | 7 | | | | | | | |
| 34 | 32 | 18 | | | | 14 | | | | | | | 6 | | | | | | | 34 |
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| 38 | | 20 | | | | | | 8 | | | | | | 2 | | | | | | 38 |
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| 43 | | 25 | | 23 | 22 | | | 13 | | 11 | 10 | | | 7 | | | | | | 43 |
| 44 | | 26 | | | | | | 14 | | 16 | − 15 | | 8 | −5 | | | | | | 44 |
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| 56 | ..... | ..... | ..... | ..... | ..... | ... | ... | ..... | ..... | ..... | ... | ..... | ..... | ... | ... | ... | ... | ... | ... | ... | 56 |
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| 57 | ..... | ..... | 35 | ..... | 33 | ... | ... | ..... | ..... | ..... | ... | ..... | ..... | 15 | ... | 3 | ... | ... | ... | ... | 57 |
| | ..... | 37 | ..... | ..... | ..... | ... | ... | ..... | ..... | ..... | ... | ..... | 19 | ... | ... | ... | ... | ... | ... | ... | |
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| 59 | ..... | ..... | ..... | ..... | 35 | ... | ... | ..... | ..... | ..... | ... | ..... | ..... | 17 | ... | 5 | ... | ... | ... | ... | 59 |
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| | ... | 157 | −156 | ..... | 151 | ..... | ..... | ..... | ..... | ..... | 147 | −146 | ..... | ..... | ... | ..... | 12 | −11 | |
| 170 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 146 | ..... | ..... | ... | ..... | 12 | ..... | 170 |
| | ... | 158 | −155 | ..... | 152 | ..... | ..... | ..... | ..... | ..... | 148 | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 171 | ... | 159 | ..... | ..... | 153 | ..... | ..... | ..... | ..... | ..... | 147 | ..... | ..... | ..... | ... | ..... | 13 | 11 | 171 |
| | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 172 | ... | 160 | ..... | 156 | 154 | ..... | ..... | ..... | ..... | ..... | 148 | ..... | ..... | ..... | ... | ..... | 13 | ..... | 172 |
| | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 173 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | −146 | ..... | ... | ..... | 14 | −15 | 173 |
| | ... | −156 | 157 | −160 | 149 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 174 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 146 | ..... | ... | ..... | 14 | ..... | 174 |
| | ... | −155 | 158 | −159 | 150 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | 16 | |
| 175 | ... | ..... | 155 | ..... | 151 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 147 | ..... | ... | ..... | ... | ..... | 175 |
| | ... | 159 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | 15 | −14 | |
| 176 | ... | ..... | 156 | ..... | 152 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 148 | ..... | ... | ..... | 15 | ..... | 176 |
| | ... | 160 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 177 | ... | 157 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 147 | ..... | ... | ..... | ... | 14 | 177 |
| | ... | ..... | ..... | −156 | 153 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | 16 | ..... | |
| 178 | ... | 158 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 148 | ..... | ... | ..... | 16 | ..... | 178 |
| | ... | ..... | ..... | −155 | 154 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 179 | ... | ..... | ... .. | ..... | 155 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | 179 |
| | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | −146 | ... | ..... | 17 | −18 | |
| 180 | ... | ..... | ..... | ..... | 156 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 146 | ... | ..... | 17 | ..... | 180 |
| | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | 19 | |
| 181 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 147 | ... | ..... | 18 | −17 | 181 |
| | ... | ..... | ..... | ..... | 157 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 182 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 148 | ... | ..... | 18 | ..... | 182 |
| | ... | ..... | ..... | ..... | 158 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 183 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 147 | ... | ..... | ... | 17 | 183 |
| | ... | ..... | ..... | ..... | 159 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | 19 | ..... | |
| 184 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | 148 | ... | ..... | ... | 19 | 184 |
| | ... | ..... | ..... | ..... | 160 | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | ..... | ... | ..... | |
| 185 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | 147 | ... | ..... | 185 |
| | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | −146 | ... | ..... | |
| 186 | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | 146 | ... | ..... | 186 |
| | ... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ..... | ... | 148 | ... | ..... | |

| | 38 | 59 | |
|---|---|---|---|
| 39 | ..... 1 | ..... | 39 |
| 40 | 1 ..... | ..... | 40 |

| | 38 | 59 | |
|---|---|---|---|
| 60 | ..... | ..... 1 | 60 |
| 61 | ..... | 1 ..... | 61 |

Table II. may be used in forming the developments required in the method employed by MM. Laplace and Damoiseau; for this purpose it is only necessary to make $t = \lambda' - \lambda_i$ instead of $nt - n_i t$

$$x = c\lambda' - \varpi \quad . \quad . \quad . \quad cnt - \varpi$$
$$z = c_i\lambda_i - \varpi_i \quad . \quad . \quad . \quad c_i n_i t - \varpi_i$$
$$\text{and } y = g\lambda' - \nu \quad . \quad . \quad . \quad gnt - \nu$$

The notation throughout is the same as that used Phil. Trans. 1830, p. 328, with the exception of the indices of the arguments.

In the elliptic movement;

$$a^5 r^{-5} = 1 + 5e^2\left(1 + \frac{21}{8}e^2\right) + 5e\left(1 + \frac{27}{8}e^2\right)\cos x + 10e^2\left(1 + \frac{31}{12}e^2\right)\cos 2x + \frac{145}{8}e^3\cos 3x + \frac{745}{48}e^4\cos 4x$$

$$a^4 r^{-4} = 1 + 3e^2 + 4e\cos x + 7e^2\cos 2x$$

$$a^3 r^{-3} = 1 + \frac{3}{2}e^2\left(1 + \frac{5}{4}e^2\right) + 3e\left(1 + \frac{9}{8}e^2\right)\cos x + \frac{9}{2}e^2\left(1 + \frac{7}{9}e^2\right)\cos 2x + \frac{53}{8}e^3\cos 3x + \frac{77}{8}e^4\cos 4x$$

$$a^2 r^{-2} = 1 + \frac{e^2}{2}\left(1 + \frac{3}{4}e^2\right) + 2e\left(1 + \frac{3}{8}e^2\right)\cos x + \frac{5}{2}e^2\left(1 + \frac{2}{15}e^2\right)\cos 2x + \frac{13}{4}e^3\cos 3x + \frac{103}{24}e^4\cos 4x$$

$$a r^{-1} = 1 + e\left(1 - \frac{e^2}{8}\right)\cos x + e^2\left(1 - \frac{e^2}{3}\right)\cos 2x + \frac{9}{8}e^3\cos 3x + \frac{4}{3}e^4\cos 4x$$

$$\frac{r}{a} = 1 + \frac{e^2}{2} - e\left(1 - \frac{3e^2}{8}\right)\cos x - \frac{e^2}{2}\left(1 - \frac{2e^2}{3}\right)\cos 2x - \frac{3e^3}{8}\cos 3x - \frac{e^4}{3}\cos 4x$$

$$\frac{r^2}{a^2} = 1 + \frac{3e^2}{2} - 2e\left(1 - \frac{e^2}{8}\right)\cos x - \frac{e^2}{2}\left(1 - \frac{e^2}{3}\right)\cos 2x - \frac{e^3}{4}\cos 3x - \frac{e^4}{6}\cos 4x$$

$$\frac{r^3}{a^3} = 1 + 3e^2\left(1 + \frac{e^2}{8}\right) - 3e\left(1 + \frac{3}{8}e^2\right)\cos x - \frac{5}{8}e^2\cos 2x + \frac{e^3}{8}\cos 3x + \frac{e^4}{8}\cos 4x$$

$$\frac{r^4}{a^4} = 1 + 5e^2 - 4e\cos x + e^2\cos 2x$$

$$\frac{a}{r} = r_0 + r_1\cos 2t + er_2\cos x + er_3\cos(2t - x) + er_4\cos(2t + x) + e_i r_5\cos z + e_i r_6\cos(2t - z) + \&c.\ \&c.$$

$$\begin{aligned}\lambda = {} & n\,t \\ & + \lambda_1 \cos 2\,t \\ & + e\,\lambda_2 \cos x \\ & + e\,\lambda_3 \cos (2\,t - x) \\ & + e\,\lambda_4 \cos (2\,t + x) \\ & + e_i\,\lambda_5 \cos z \ \&c.\ \&c.\end{aligned}$$

The quantities $\lambda$ correspond to the quantities $b$ in M. Damoiseau's notation.

$$\begin{aligned}s = {} & \gamma\, s_{148} \sin y \\ & + \gamma\, s_{147} \sin (2\,t - y) \\ & + \gamma\, s_{149} \sin (2\,t + y) \\ & + e\gamma\, s_{149} \sin (x - y) \ \&c.\ \&c. \\ & \gamma = \tan s\end{aligned}$$

$$R = m_i \left\{ \frac{r' r_i \cos (\lambda - \lambda_i)}{r_i^3} - \frac{1}{\{r'^2 - 2\,r' r_i \cos (\lambda' - \lambda_i) + r_i^2\}^{\frac{1}{2}}} \right\}$$

$$= m_i \left\{ -\frac{1}{r_i} + \frac{r'^2}{2\,r_i^3} - \frac{3}{8} \frac{\{2\,r' r_i \cos (\lambda' - \lambda_i) - r'^2\}^2}{r_i^5} - \frac{15}{48} \frac{\{2\,r' r_i \cos (\lambda' - \lambda_i) - r'^2\}^3}{r_i^7} \right\}$$

$$= m_i \left\{ -\frac{1}{r_i} + \frac{r'^2}{2\,r_i^3} - \frac{3}{2} \frac{r'^2 r_i^2}{r_i^5} \cos (\lambda' - \lambda_i)^2 + \frac{3}{2} \frac{r'^2 r' r_i}{r_i^5} \cos (\lambda - \lambda_i) - \frac{5}{2} \frac{r'^3 r_i^3}{r_i^7} \cos (\lambda' - \lambda_i)^3 \right\}$$

$$= m_i \left\{ -\frac{1}{r_i} - \frac{r'^2}{4\,r_i^3} \left\{ 1 + 3 \cos (2\,\lambda' - 2\,\lambda_i) - 2\,s^2 \right\} \right.$$
$$\left. - \frac{r'^3}{8\,r_i^4} \left\{ 3\,(1 - 4\,s^2) \cos (\lambda' - \lambda_i) + 5 \cos (3\,\lambda' - 3\,\lambda_i) \right\} \right\}$$

$$r' r_i \cos (\lambda' - \lambda_i) = r r_i \left\{ \cos^2 \tfrac{i}{2} \cos (\lambda - \lambda_i) + \sin^2 \tfrac{i}{2} \cos (\lambda + \lambda_i - 2\nu) \right\}$$

$$= {}^{*}\, a\, a_i \cos^2 \tfrac{i}{2} \left\{ \left( 1 - \frac{e^2}{2} - \frac{e^4}{64} \right) \left( 1 - \frac{e_i^2}{2} - \frac{e_i^4}{64} \right) \cos \sin t - \frac{3\,e}{2} \left( 1 - \frac{e_i^2}{2} \right) \cos \sin (t - x) \right.$$
$$+ \frac{e}{2} \left( 1 - \frac{3}{4} e^2 \right) \left( 1 - \frac{e_i^2}{2} \right) \cos \sin (t + x) + \frac{3}{8} e^2 (1 - e^2) \left( 1 - \frac{e_i^2}{2} \right) \cos \sin (t + 2x)$$
$$+ \frac{e^3}{3} \cos \sin (t + 3x) + \frac{125}{384} e^4 \cos \sin (t + 4x) + \frac{e^2}{8} \left( 1 + \frac{e^2}{3} \right) \left( 1 - \frac{e_i^2}{2} \right) \cos \sin (t - 2x)$$
$$+ \frac{e^3}{24} \cos \sin (t - 3x) + \frac{3}{128} e^4 \cos \sin (t - 4x) - \frac{3}{2} e_i \left( 1 - \frac{e^2}{2} \right) \cos \sin (t + z)$$
$$+ \frac{9}{4} e\, e_i \cos \sin (t - x + z) - \frac{3}{4} e\, e_i \left( 1 - \frac{3}{4} e^2 \right) \cos \sin (t + x + z)$$
$$\left. - \frac{9}{16} e^2 e_i \cos \sin (t + 2x + z) - \frac{e^3 e_i}{2} \cos \sin (t + 3x + z) - \frac{3}{16} e^2 e_i \cos \sin (t - 2x + z) \right.$$

* See Phil. Trans. 1830, p. 343.

$$-\frac{e^3 e_i}{16}\frac{\cos}{\sin}(t-3x+z)+\frac{e_i}{2}\left(1-\frac{3}{4}e_i^2\right)\left(1-\frac{e^2}{2}\right)\frac{\cos}{\sin}(t-z)$$

$$-\frac{3}{4}e e_i\left(1-\frac{3}{4}e_i^2\right)\frac{\cos}{\sin}(t-x-z)$$

$$+\frac{e e_i}{4}\left(1-\frac{3}{4}e^2\right)\left(1-\frac{3}{4}e_i^2\right)\frac{\cos}{\sin}(t+x-z)+\frac{3}{16}e^2 e_i\frac{\cos}{\sin}(t+2x-z)$$

$$+\frac{e^3 e_i}{6}\frac{\cos}{\sin}(t+3x-z)+\frac{e^2 e_i}{16}\frac{\cos}{\sin}(t-2x-z)+\frac{e^3 e_i}{48}\frac{\cos}{\sin}(t-3x-z)$$

$$+\frac{3}{8}e_i^2(1-e_i^2)\left(1-\frac{e^2}{2}\right)\frac{\cos}{\sin}(t-2z)-\frac{9}{16}e e_i^2\frac{\cos}{\sin}(t-x-2z)$$

$$+\frac{3}{16}e e_i^2\frac{\cos}{\sin}(t+x-2z)+\frac{9}{64}e^2 e_i^2\frac{\cos}{\sin}(t+2x-2z)$$

$$+\frac{3}{64}e^3 e_i^2\frac{\cos}{\sin}(t-2x-2z)+\frac{e_i^3}{3}\frac{\cos}{\sin}(t-3z)-\frac{e e_i^3}{2}\frac{\cos}{\sin}(t-x-3z)$$

$$+\frac{e e_i^3}{6}\frac{\cos}{\sin}(t+x-3z)+\frac{125}{384}e_i^4\frac{\cos}{\sin}(t-4z)$$

$$+\frac{e_i^2}{8}\left(1+\frac{e_i^2}{3}\right)\left(1-\frac{e^2}{2}\right)\frac{\cos}{\sin}(t+2z)-\frac{3}{16}e e_i^2\frac{\cos}{\sin}(t-x+2z)$$

$$+\frac{e e_i^2}{16}\frac{\cos}{\sin}(t+x+2z)+\frac{3}{64}e^2 e_i^2\frac{\cos}{\sin}(t+2x+2z)+\frac{e^2 e_i^2}{64}\frac{\cos}{\sin}(t-2x+2z)$$

$$+\frac{e_i^3}{24}\frac{\cos}{\sin}(t+3z)-\frac{e e_i^3}{16}\frac{\cos}{\sin}(t-x+3z)+\frac{e e_i^3}{48}\frac{\cos}{\sin}(t+x+3z)$$

$$+\frac{3}{128}e_i^4\frac{\cos}{\sin}(t+4z)\Big\}$$

$$+aa_i\sin^2\frac{i}{2}\left\{\left(1-\frac{e^2+e_i^2}{2}\right)\frac{\cos}{\sin}(t-2y)-\frac{3}{2}e\frac{\cos}{\sin}(t+x-2y)+\frac{e}{2}\frac{\cos}{\sin}(t-x-2y)\right.$$

$$+\frac{3}{8}e^2\frac{\cos}{\sin}(t-2x-2y)+\frac{e^2}{8}\frac{\cos}{\sin}(t+2x-2y)-\frac{3}{2}e_i\frac{\cos}{\sin}(t+z-2y)$$

$$+\frac{9}{4}e e_i\frac{\cos}{\sin}(t+x+z-2y)-\frac{3}{4}e e_i\frac{\cos}{\sin}(t-x+z-2y)$$

$$\left.+\frac{e_i}{2}\frac{\cos}{\sin}(t-z-2y)-\frac{3}{4}e e_i\frac{\cos}{\sin}(t+x-z-2y)+\frac{e e_i}{4}\frac{\cos}{\sin}(t-x-z-2y)\right.$$

$$r'^2 r_i^2\cos(\lambda'-\lambda_i)^2$$

$$=a^2 a_i^2\cos^4\frac{i}{2}\left\{\frac{1}{2}+\left\{-\frac{1}{2}+\frac{9}{8}+\frac{1}{8}\right\}(e^2+e_i^2)+\left\{\frac{1}{2}-\frac{9}{8}-\frac{1}{8}-\frac{9}{8}+\frac{81}{32}+\frac{9}{32}\right.\right.$$

$$\left.\left.-\frac{1}{8}+\frac{9}{32}+\frac{1}{32}\right\}e^2 e_i^2+\left\{\frac{7}{64}-\frac{3}{16}+\frac{9}{128}+\frac{1}{128}\right\}(e^4+e_i^4)\right.$$

[0]

$$+\left\{\frac{1}{2}+\left\{-\frac{1}{2}-\frac{3}{4}\right\}(e^2+e_1^2)+\left\{\frac{1}{2}+\frac{3}{4}+\frac{3}{4}+\frac{9}{16}+\frac{9}{16}\right\}e^2e_1^2\right.$$
$$\left.+\left\{\frac{7}{64}+\frac{9}{16}+\frac{3}{64}\right\}(e^4+e_1^4)\right\}\cos 2t$$
[1]

$$+\left\{-\frac{3}{2}+\frac{1}{2}+\left\{\frac{3}{4}-\frac{5}{8}-\frac{3}{16}+\frac{3}{16}\right\}e^2\right.$$
$$\left.+\left\{\frac{3}{2}-\frac{1}{2}-\frac{27}{8}+\frac{9}{8}-\frac{3}{8}+\frac{1}{8}\right\}e_1^2\right\}e\cos x$$
[2] [5]*

$$+\left\{-\frac{3}{2}+\left\{\frac{3}{4}+\frac{1}{16}\right\}e^2+\left\{\frac{3}{2}+\frac{9}{8}+\frac{9}{8}\right\}e_1^2\right\}e\cos(2t-x)$$
[3] [7]

$$+\left\{-\frac{1}{2}+\left\{\frac{5}{8}-\frac{9}{16}\right\}e^2+\left\{-\frac{1}{2}-\frac{3}{8}-\frac{3}{8}\right\}e_1^2\right\}e\cos(2t+x)$$
[4] [6]

$$+\left\{\frac{3}{8}+\frac{1}{8}-\frac{3}{4}+\left\{-\frac{9}{16}-\frac{1}{48}+\frac{9}{16}-\frac{1}{16}+\frac{1}{6}\right\}e^2\right.$$
$$\left.+\left\{-\frac{3}{8}-\frac{1}{8}+\frac{3}{4}+\frac{27}{32}+\frac{9}{32}-\frac{27}{16}+\frac{3}{32}+\frac{1}{32}-\frac{3}{16}\right\}e_1^2\right\}e^2\cos 2x$$
[8] [17]

$$+\left\{\frac{9}{8}+\frac{1}{8}+\left\{-\frac{1}{48}+\frac{1}{48}\right\}e^2\right.$$
$$\left.+\left\{-\frac{9}{8}-\frac{1}{8}-\frac{3}{32}-\frac{27}{16}-\frac{3}{32}\right\}e_1^2\right\}e^2\cos(2t-2x)$$
[9] [19]

$$+\left\{\frac{1}{8}+\frac{3}{8}+\left\{-\frac{3}{16}-\frac{9}{16}-\frac{1}{2}\right\}e^2\right.$$
$$\left.+\left\{-\frac{1}{8}-\frac{3}{8}-\frac{9}{32}-\frac{3}{16}-\frac{9}{32}\right\}e_1^2\right\}e^2\cos(2t+2x)$$
[10] [18]

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\frac{9}{4}+\frac{1}{4}+\left\{\frac{15}{16}+\frac{3}{8}-\frac{9}{8}-\frac{3}{32}-\frac{9}{32}\right.\right.\quad\frac{5}{16}$$
$$\left.\left.+\frac{3}{32}+\frac{9}{32}\right\}(e^2+e_1^2)\right\}ee_1\cos(x+z)$$
[11]

* The coefficient of argument 5 being the same, $e$ and $e_1$ changing places, that coefficient is not written down, in order to avoid useless repetition.

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\left\{\frac{3}{8}+\frac{3}{8}+\frac{1}{32}+\frac{1}{32}\right\}e^2\right.$$
$$\left.+\left\{\frac{15}{16}+\frac{15}{16}+\frac{27}{32}+\frac{27}{32}\right\}e_i^2\right\}e e_i\cos(2t-x-z)$$
[12] [13]

$$+\left\{\frac{9}{4}+\frac{1}{4}\quad\frac{3}{4}-\frac{3}{4}+\left\{-\frac{9}{8}-\frac{5}{16}+\frac{9}{32}+\frac{3}{8}+\frac{15}{16}\right.\right.$$
$$\left.\left.+\frac{3}{32}-\frac{9}{32}-\frac{3}{32}\right\}(e^2+e_i^2)\right\}e e_i\cos(x-z)$$
[14]

$$+\left\{\frac{9}{4}+\frac{9}{4}+\left\{-\frac{9}{8}-\frac{9}{8}-\frac{3}{32}-\frac{3}{32}\right\}(e^2+e_i^2)\right\}e e_i\cos(2t-x+z)$$
[15]

$$+\left\{\frac{1}{4}+\frac{1}{4}+\left\{-\frac{5}{16}-\frac{9}{32}-\frac{5}{16}-\frac{9}{32}\right\}(e_i+e_i^2)\right\}\cos(2t+x-z)$$
[16]

$$+\left\{\frac{1}{3}+\frac{1}{24}-\frac{9}{16}+\frac{1}{16}\right\}e\cos 3x$$
[20] [35]

$$+\left\{\frac{1}{24}-\frac{3}{16}\right\}e^3\cos(2t-3x)$$
[21] [37]

$$+\left\{\frac{1}{3}+\frac{3}{16}\right\}e^3\cos(2t+3x)$$
[22] [36]

$$+\left\{-\frac{9}{16}+\frac{1}{16}+\frac{9}{8}-\frac{3}{8}+\frac{3}{16}-\frac{3}{16}\right\}e^2e_i\cos(2x+z)$$
[23] [29]

$$+\left\{\frac{1}{16}+\frac{9}{8}+\frac{1}{16}\right\}e^2e_i\cos(2t-2x-z)$$
[24] [31]

$$+\left\{-\frac{9}{16}-\frac{3}{8}-\frac{9}{16}\right\}e^2e_i\cos(2t+2x+z)$$
[25] [30]

$$+\left\{-\frac{3}{16}+\frac{3}{16}-\frac{3}{8}+\frac{9}{8}-\frac{9}{16}+\frac{1}{16}\right\}e^2e_i\cos(2x-z)$$
[26] [32]

$$+\left\{-\frac{3}{16}-\frac{27}{8}-\frac{3}{16}\right\}e^2e_i\cos(2t-2x+z)$$
[27] [33]

$$+\left\{\frac{3}{16}+\frac{1}{8}+\frac{3}{16}\right\}e^2e_i\cos(2t+2x-z)$$
[28] [34]

$$+\left\{\frac{125}{384}+\frac{3}{128}-\frac{1}{2}+\frac{1}{48}+\frac{3}{64}\right\}e^4\cos 4x$$
[38] [59]

$$+\left\{\frac{1}{128}+\frac{3}{128}-\frac{1}{16}\right\}e^4\cos(2t-4x)$$
[39] [61]

$$+\left\{\frac{9}{128}+\frac{125}{384}+\frac{1}{6}\right\}e^4\cos(2t+4x)$$
[40] [60]

$$+\left\{-\frac{1}{2}+\frac{1}{48}+\frac{27}{32}+\frac{1}{32}-\frac{9}{32}+\frac{1}{6}-\frac{3}{32}-\frac{1}{16}\right\}e^3\cos(3x+z)$$
[41] [53]

$$+\left\{\frac{1}{48}-\frac{3}{32}-\frac{3}{32}+\frac{1}{48}\right\}e^3e_i\cos(2t-3x-z)$$
[42] [55]

$$+\left\{-\frac{1}{2}-\frac{9}{32}-\frac{9}{32}-\frac{1}{2}\right\}e^3e_i\cos(2t+3x+z)$$
[43] [54]

$$+\left\{-\frac{1}{16}+\frac{1}{6}-\frac{9}{32}-\frac{3}{32}+\frac{27}{32}-\frac{1}{2}+\frac{1}{32}+\frac{1}{48}\right\}e^3e_i\cos(3x-z)$$
[44] [56]

$$+\left\{-\frac{1}{16}+\frac{9}{32}+\frac{9}{32}-\frac{1}{16}\right\}e^3e_i\cos(2t-3x+z)$$
[45] [57]

$$+\left\{\frac{1}{6}+\frac{3}{32}+\frac{3}{32}+\frac{1}{6}\right\}e^3e_i\cos(2t+3x-z)$$
[46] [58]

$$+\left\{\frac{3}{64}+\frac{3}{64}-\frac{3}{32}-\frac{9}{32}+\frac{9}{64}+\frac{1}{64}-\frac{3}{32}+\frac{9}{16}-\frac{9}{32}\right\}e^3e_i^2\cos(2x+2z)$$
[47]

$$+\left\{\frac{9}{32}+\frac{3}{64}+\frac{27}{32}+\frac{3}{64}+\frac{1}{32}\right\}e^3e_i^2\cos(2t-2x+2z)$$
[48]

$$+\left\{\frac{9}{32}+\frac{3}{64}+\frac{1}{32}+\frac{3}{64}+\frac{27}{32}\right\}e^3e_i^2\cos(2t+2x+2z)$$
[49]

$$+\left\{\frac{9}{64}+\frac{1}{64}-\frac{9}{32}-\frac{3}{32}+\frac{3}{64}+\frac{3}{64}-\frac{9}{32}+\frac{9}{16}-\frac{3}{32}\right\}e^3e_i^2\cos(2x-2z)$$
[50]

$$+\left\{\frac{81}{32}+\frac{1}{64}+\frac{9}{32}+\frac{1}{64}+\frac{9}{32}\right\}e^3e_i^2\cos(2t-2x+2z)$$
[51]

$$+\left\{\frac{1}{32}+\frac{9}{64}+\frac{3}{32}+\frac{9}{64}+\frac{3}{32}\right\}e^3e_i^2\cos(2t+2x-2z)$$
[52]

$$+a^2a_i^2\sin^2\tfrac{i}{2}\cos^2\tfrac{i}{2}\left\{\left\{1+\left\{-1-\frac{3}{4}-\frac{3}{4}\right\}e^2+\left\{-1+\frac{9}{4}+\frac{1}{4}\right\}e_i^2\right\}\cos 2y\right.$$
[62]

$$\left.+\left\{1+\left\{-1+\frac{9}{4}+\frac{1}{4}\right\}e^2+\left\{-1-\frac{3}{4}-\frac{3}{4}\right\}e_i^2\right\}\cos(2t-2y)\right.$$
[63]

$$+\left\{-\frac{3}{2}-\frac{3}{2}\right\}e\cos(x-2y)+\left\{\frac{1}{2}+\frac{1}{2}\right\}e\cos(x+2y)$$
[65] [66]

$$+\left\{\frac{1}{2}-\frac{3}{2}\right\}e\cos(2t-x-2y)$$
[67]

$$+\left\{-\frac{3}{2}+\frac{1}{2}\right\}e\cos(2t+x-2y)+\left\{-\frac{3}{2}+\frac{1}{2}\right\}e_1\cos(x-2y)$$
[69] [71]

$$+\left\{\frac{1}{2}-\frac{3}{2}\right\}e_1\cos(x+2y)+\left\{\frac{1}{2}+\frac{1}{2}\right\}e_1\cos(2t-x-2y)$$
[72] [73]

$$+\left\{-\frac{3}{2}-\frac{3}{2}\right\}e_1\cos(2t+x-2y)+\left\{\frac{1}{8}+\frac{9}{4}+\frac{1}{8}\right\}e^2\cos(2x-2\ )$$
[75] [77]

$$+\left\{\frac{3}{8}+\frac{1}{4}+\frac{3}{8}\right\}e^2\cos(2x+2y)$$
[78]

$$+\left\{\frac{3}{8}-\frac{3}{4}+\frac{1}{8}\right\}e^2\cos(2t-2x-2y)+\left\{\frac{1}{8}-\frac{3}{4}+\frac{3}{8}\right\}e^2\cos(2t+2x-2y)$$
[79] [81]

$$+\left\{\frac{9}{4}+\frac{9}{4}-\frac{3}{4}-\frac{3}{4}\right\}ee_1\cos(x+z-2y)$$
[83]

$$+\left\{\frac{1}{4}+\frac{1}{4}-\frac{3}{4}-\frac{3}{4}\right\}ee_1\cos(x+z+2y)$$
[84]

$$+\left\{\frac{1}{4}-\frac{3}{4}+\frac{1}{4}-\frac{3}{4}\right\}ee_1\cos(2t-x-z-2y)$$
[85]

$$+\left\{\frac{9}{4}-\frac{3}{4}+\frac{9}{4}-\frac{3}{4}\right\}ee_1\cos(2t+x+z+2y)$$
[87]

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\frac{9}{4}+\frac{9}{4}\right\}ee_1\cos(x-z-2y)$$
[89]

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\frac{1}{4}+\frac{1}{4}\right\}ee_1\cos(x-z+2y)$$
[90]

$$+\left\{-\frac{3}{4}+\frac{9}{4}-\frac{3}{4}+\frac{9}{4}\right\}ee_1\cos(2t-x+z-2y$$
[91]

$$+\left\{-\frac{3}{4}+\frac{1}{4}-\frac{3}{4}+\frac{1}{4}\right\}ee_1\cos(2t+x-z-2y$$
[93]

$$+\left\{-\frac{3}{4}+\frac{3}{8}\right\}e_1^2\cos(2z-2y)+\left\{-\frac{3}{4}+\frac{1}{8}\right\}e_1^2\cos(2z+2y)$$
[95] [96]

$$+\left\{\frac{1}{4}+\frac{3}{8}\right\}e_1^2\cos(2t-2z-2y)+\left\{\frac{9}{4}+\frac{1}{8}\right\}e_1^2\cos(2t+2z-2y)$$
[97] [99]

$$+a^3a_1^3\sin^4\frac{i}{2}\left\{\frac{1}{2}+\frac{1}{2}\cos(2t-2y)\right\}$$
[63]

$$r'^3r^3\cos(\lambda'-\lambda)^3$$

$$=a^3a_1^3\cos^4\frac{i}{2}\left\{\frac{1}{2}+\frac{3}{4}(e^2+e_1^2)+\frac{9}{8}e^2e_1^2+\left\{\frac{1}{2}-\frac{5}{4}(e^2+e_1^2)\right.\right.$$

$$\left.\left.+\frac{23}{32}(e^4+e_1^4)+\frac{25}{8}e^2e_1^2\right\}\cos 2t+\left\{-1+\frac{e^2}{8}-\frac{3}{2}e_1^2\right\}e\cos x\right.$$
[1] [2] [5]

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{15}{4}e_1^2\right\}e\cos(2t-x)+\left\{\frac{1}{2}-\frac{19}{16}e^2-\frac{5}{4}e_1^2\right\}e\cos(2t+x)$$
[3] [7] [4] [6]

$$+\left\{-\frac{1}{4}+\frac{1}{12}e^2-\frac{3}{8}e_1^2\right\}e^2\cos 2x$$
[8] [17]

$$+\left\{\frac{5}{4}-\frac{25}{8}e_1^2\right\}e^2\cos(2t-2x)+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_1^2\right\}e\cos(2t+2x)$$
[9] [19] [10] [18]

$$+\left\{1-\frac{1}{8}(e^2+e_1^2)\right\}ee_1\cos(x+z)$$
[11]

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{57}{16}e_1^2\right\}ee_1\cos(2t-x-z)$$
[12] [13]

$$+\left\{1-\frac{(e^2+e_1^2)}{8}\right\}ee_1\cos(x-z)$$
[14]

$$+\left\{\frac{9}{2}-\frac{39}{16}(e^2+e_1^2)\right\}ee_1\cos(2t-x+z)$$
[15]

$$+\left\{\frac{1}{2}-\frac{19}{16}(e^2+e_1^2)\right\}ee_1\cos(2t+x-z)-\frac{e^2}{8}\cos 3x$$
[16] [20] [35]

$$-\frac{7e^3}{48}\cos(2t-3x)+\frac{25e^3}{48}\cos(2t+3x)+\frac{e^2e_1}{4}\cos(2x+z)$$
[21] [37] [22] [36] [23] [29]

$$+\frac{5}{4}e^3e_1\cos(2t-2x-z)-\frac{25}{16}e^2e_1\cos(2t+2x+z)+\frac{e^2e_1}{4}\cos(2x-z)$$
[24] [31] [25] [30] [26] [32]

$$-\frac{15}{4}e^2e_1\cos(2t-2x+z)+\frac{e^2e_1}{2}\cos(2t+2x-z)$$
[27] [33] [28] [34]

$$-\frac{e^4}{12}\cos 4x-\frac{e^4}{32}\cos(2t-4x)+\frac{9}{16}e^4\cos(2t+4x)+\frac{e^3e_1}{8}\cos(3x+z)$$
[38] [59] [39] [61] [40] [60] [41] [53]

$$-\frac{7}{48}e^3e_1\cos(2t-3x-z)-\frac{25}{16}e^2e_1\cos(2t+3x+z)$$
[42] [55] [43] [54]

$$+\frac{e^3e_1}{8}\cos(3x-z)+\frac{7}{16}e^3e_1\cos(2t-3x+z)+\frac{25}{48}e^3e_1\cos(2t+3x-z)$$
[44] [56] [45] [57] [46] [58]

$$+\frac{e^3e_1}{16}\cos(2x+2z)+\frac{5}{4}e^3e_1^3\cos(2t-2x-2z)$$
[47] [48]

$$+\frac{5}{4}e^2e_1\cos(2t+2x+2z)+\frac{e^3e_1^3}{16}\cos(2x-2z)$$
[49] [50]

$$+\frac{25}{8}e^3e_1^3\cos(2t-2x+2z)+\frac{e^2e_1^3}{2}\cos(2t+2x-2z)$$
[51] [52]

$$+a^3a_1^3\cos^2\frac{i}{2}\sin^2\frac{i}{2}\left\{\left\{1-\frac{5}{2}e^2+\frac{3}{2}e_1^2\right\}\cos 2y+\left\{1+\frac{3}{2}e^2-\frac{5}{2}e_1^2\right\}\cos(2t-2y)\right.$$
[62] [63]

$$-3e\cos(x-2y)+e\cos(x+2y)-e\cos(2t-x-2y)$$
[65] [66] [67]

$$-e\cos(2t+x-2y)-e_1\cos(x-2y)-e_1\cos(z+2y)$$
[69] [71] [72]

$$+e_1\cos(2t-z-2y)-3e_1\cos(2t+z-2y)$$
[73] [75]

$$+\frac{5}{2}e^2\cos(2x-2y)+e^2\cos(2x+2y)$$
[77] [78]

$$-\frac{e^2}{4}\cos(2t-2x-2y)-\frac{e^2}{4}\cos(2t+2x-2y)$$
[79] [81]

$$+ 3\,e\,e_{,}\cos(x+z-2y) \quad [83]$$
$$- e\,e_{,}\cos(x+z+2y) \quad [84]$$
$$- e\,e_{,}\cos(2t-x-z-2y) \quad [85]$$
$$+ 3\,e\,e_{,}\cos(2t+x+z-2y) \quad [87]$$
$$+ 3\,e\,e_{,}\cos(x-z-2y) \quad [89]$$
$$- e\,e_{,}\cos(x-z+2y) \quad [90]$$
$$+ 3\,e\,e_{,}\cos(2t-x+z-2y) \quad [91]$$
$$- e\,e_{,}\cos(2t+x-z-2y) \quad [93]$$
$$- \frac{3}{8}\,e_{,}^{2}\cos(2x-2y) \quad [95]$$
$$- \frac{5}{8}\,e_{,}^{2}\cos(2x+2y) \quad [96]$$
$$+ \frac{5\,e_{,}^{2}}{8}\cos(2t-2x-2y) \quad [97]$$
$$+ \frac{19}{8}\,e_{,}^{2}\cos(2t+2x-2y) \quad [99]$$
$$\Big\}$$
$$+ a^{6}\,a_{,}^{-6}\sin^{4}\frac{i}{2}\left\{\frac{1}{2}+\frac{1}{2}\cos(2t-2y)\right\} \quad [68]$$

$$\frac{r^{3}}{2\,r_{,}^{3}} = \frac{a^{3}}{a_{,}^{3}}\left\{\frac{1}{2}+\frac{3}{4}e^{2}+\frac{3}{4}e_{,}^{2}+\frac{9}{8}e^{2}e_{,}^{2}+\frac{15}{16}e_{,}^{4}-e\left\{1-\frac{e^{2}}{8}+\frac{3}{2}e_{,}^{2}\right\}\cos x\right. \quad [2]$$
$$+ \frac{3}{2}e_{,}\left\{1+\frac{3}{2}e^{2}+\frac{9}{8}e_{,}^{2}\right\}\cos z - \frac{e^{2}}{4}\left\{1-\frac{e^{2}}{3}+\frac{3\,e_{,}^{2}}{2}\right\}\cos 2x \quad [5] \quad [8]$$
$$- \frac{3}{2}e\,e_{,}\left\{1-\frac{e^{2}}{8}+\frac{9}{8}e_{,}^{2}\right\}\cos(x+z) \quad [11]$$
$$- \frac{3}{2}e\,e_{,}\left\{1-\frac{e^{2}}{8}+\frac{9}{8}e_{,}^{2}\right\}\cos(x-z) \quad [14]$$
$$+ \frac{9}{4}e_{,}^{2}\left\{1+\frac{7}{9}e_{,}^{2}+\frac{3}{2}e^{2}\right\}\cos 2z - \frac{e^{3}}{8}\cos 3x - \frac{3}{8}e^{2}e_{,}\cos(2x+z) \quad [17] \quad [20] \quad [23]$$
$$- \frac{3}{8}e^{2}e_{,}\cos(2x-z) - \frac{9}{4}e\,e_{,}^{2}\cos(x+2z) - \frac{9}{4}e\,e_{,}^{2}\cos(x-2z) \quad [26] \quad [29] \quad [32]$$
$$+ \frac{53}{16}e_{,}^{3}\cos 3z - \frac{e^{4}}{12}\cos 4x - \frac{3\,e^{3}e_{,}}{16}\cos(3x+z) \quad [35] \quad [38] \quad [41]$$

$$-\frac{3}{16}e^3 e_1 \cos(3x-z) - \frac{9}{16}e^2 e_1^2 \cos(2x+2z) - \frac{9}{16}e^2 e_1 \cos(2x-2z)$$
[44] [47] [50]

$$-\frac{53}{16}e e_1^2 \cos(z+3z) - \frac{53}{16}e e_1^2 \cos(x-3z) + \frac{77}{16}e_1^3 \cos 4x$$
[53] [56] [59]

Terms in $R$ multiplied by $-\frac{3}{2}\cos^4\frac{i}{2}\frac{a^2}{a_1^2}$.

$$= * \left\{\frac{1}{2} + \frac{3}{4}(e^2+e_1^2) + \frac{9}{8}e^2e_1^2\right\}\left\{1 + 5e_1^2 + \frac{105}{8}e_1^4\right\}$$
[0]

$$+ \left\{-1 + \frac{e_1^2}{8} - \frac{3}{2}e^2\right\}\left\{\frac{5}{2}e_1^2 + \frac{135}{16}e_1^4\right\} - \frac{5}{4}e_1^4$$

$$+ \left\{\left\{\frac{1}{2} - \frac{5}{4}(e^2+e_1^2) + \frac{23}{32}(e^4+e_1^4) + \frac{25}{8}e^2e_1^2\right\}\left\{1 + 5e_1^2 + \frac{105}{8}e_1^4\right\}\right.$$

$$+ \left\{\frac{1}{2} - \frac{19}{16}e_1^2 - \frac{5}{4}e^2 - \frac{3}{2} + \frac{13}{16}e_1^2 + \frac{15}{4}e^2\right\}\left\{\frac{5}{2}e_1^2 + \frac{135}{16}e_1^4\right\}$$

$$\left. + \left\{\frac{5}{2} + \frac{25}{4}\right\}e_1^4\right\}\cos 2t$$
[1]

$$+ \left\{-1 + \frac{e^2}{8} - \frac{3}{2}e_1^2 - 5e_1^2 + \frac{5}{2}e_1^2 + \frac{5}{2}e_1^2\right\}e\cos x$$
[2]

$$+ \left\{-\frac{3}{2} + \frac{13}{16}e^2 + \frac{15}{4}e_1^2 - \frac{15}{2}e_1^2 - \frac{15}{4}e_1^2 + \frac{45}{4}e_1^2\right\}e\cos(2t-x)$$
[3]

$$+ \left\{\frac{1}{2} - \frac{19}{16}e^2 - \frac{5}{4}e_1^2 + \frac{5}{2}e_1^2 + \frac{5}{4}e_1^2 - \frac{15}{4}e_1^2\right\}e\cos(2t+x)$$
[4]

$$+ \left\{-1 + \frac{e_1^2}{8} - \frac{3}{2}e^2 - 5e_1^2 + \frac{5}{2} + \frac{15}{4}e^2 + \frac{15}{4}e_1^2 + \frac{135}{16}e_1^2 - \frac{5}{8}e_1^2 - 5e_1^2\right\}e_1\cos x$$
[5]

$$+ \left\{\frac{1}{2} - \frac{19}{16}e_1^2 - \frac{5}{4}e^2 + \frac{5}{2}e_1^2 + \frac{5}{4}e_1^2 + \frac{5}{4} - \frac{25}{8}e^2 - \frac{25}{8}e_1^2\right.$$

$$\left. + \frac{135}{32}e_1^2 - \frac{15}{2}e_1^2\right\}e_1\cos(2t-x)$$
[6]

* This multiplication of $r^2 r_1^2 \cos(\lambda - \lambda_1)^2$ by $r_1^3$ may be effected at once by means of Table II.

$$+\left\{-\frac{3}{2}+\frac{13}{16}e_{,}^{2}+\frac{15}{4}e^{2}-\frac{15}{2}e_{,}^{2}+\frac{5}{4}-\frac{25}{8}e^{2}-\frac{25}{8}e_{,}^{2}+\frac{135}{32}e_{,}^{2}+\frac{25}{8}e_{,}^{2}+\frac{5}{2}e_{,}^{2}\right\}e_{,}\cos(2t+x)$$

[7]

$$+\left\{-\frac{1}{4}+\frac{e^{2}}{12}-\frac{3}{8}e_{,}^{2}-\frac{5}{4}e_{,}^{2}+\frac{5}{8}e_{,}^{2}+\frac{5}{8}e_{,}^{2}\right\}e^{2}\cos 2x$$

[8]

$$+\left\{\frac{5}{4}-\frac{25}{8}e_{,}^{2}+\frac{25}{4}e_{,}^{2}+\frac{25}{8}e_{,}^{2}-\frac{75}{8}e_{,}^{2}\right\}e^{2}\cos(2t-2x)$$

[9]

$$+\left\{\frac{1}{2}-\frac{5}{4}e^{2}-\frac{5}{4}e_{,}^{2}+\frac{5}{2}e_{,}^{2}+\frac{5}{4}e_{,}^{2}-\frac{15}{4}e_{,}^{2}\right\}e^{2}\cos(2t+2x)$$

[10]

$$+\left\{1-\frac{e^{2}}{8}-\frac{e_{,}^{2}}{8}+5e_{,}^{2}-\frac{5}{2}+\frac{5e^{2}}{16}-\frac{15}{4}e_{,}^{2}-\frac{135}{16}e_{,}^{2}+\frac{5e_{,}^{2}}{8}+5e_{,}^{2}\right\}ee_{,}\cos(x+z)$$

[11]

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^{2}+\frac{57}{16}e_{,}^{2}-\frac{15}{2}e_{,}^{2}-\frac{15}{4}e_{,}^{2}-\frac{15}{4}+\frac{65}{32}e^{2}+\frac{75}{8}e_{,}^{2}\right.$$

$$\left.-\frac{405}{32}e_{,}^{2}+\frac{45}{2}e_{,}^{2}\right\}ee_{,}\cos(2t\quad x-z)+\left\{-\frac{3}{2}+\frac{13}{16}e_{,}^{2}+\frac{57}{16}e^{2}-\frac{15}{2}e_{,}^{2}+\frac{5}{4}\right.$$

[12]

$$\left.-\frac{95}{32}e^{2}-\frac{25}{8}e_{,}^{2}+\frac{135}{32}e_{,}^{2}+\frac{25}{8}e_{,}^{2}+\frac{5}{2}e_{,}^{2}\right\}ee_{,}\cos(2t+x+z)$$

[13]

$$+\left\{1-\frac{e^{2}}{8}-\frac{e_{,}^{2}}{8}+5e_{,}^{2}+\frac{5e_{,}^{2}}{8}-\frac{5}{2}+\frac{5}{16}e^{2}-\frac{15}{4}e_{,}^{2}-\frac{135}{16}e_{,}^{2}+5e_{,}^{2}\right\}ee_{,}\cos(x-z)$$

[14]

$$+\left\{\frac{9}{2}-\frac{39}{16}e^{2}-\frac{39}{16}e_{,}^{2}+\frac{45}{2}e_{,}^{2}-\frac{15}{4}+\frac{65}{32}e^{2}+\frac{75}{8}e_{,}^{2}-\frac{405}{32}e_{,}^{2}\right.$$

$$\left.-\frac{75}{8}e_{,}^{2}-\frac{15}{2}e_{,}^{2}\right\}ee_{,}\cos(2t-x+z)$$

[15]

$$+\left\{\frac{1}{2}-\frac{19}{16}e^{2}-\frac{19}{16}e_{,}^{2}+\frac{5}{2}e_{,}^{2}+\frac{5}{4}e_{,}^{2}+\frac{5}{4}-\frac{95}{32}e^{2}-\frac{25}{8}e_{,}^{2}+\frac{135}{32}e_{,}^{2}-\frac{15}{2}e_{,}^{2}\right\}ee_{,}\cos(2t+x-z)$$

[16]

$$+\left\{-\frac{1}{4}+\frac{e_{,}^{2}}{12}-\frac{3}{8}e^{2}-\frac{5}{4}e_{,}^{2}-\frac{5}{2}+\frac{5}{16}e_{,}^{2}-\frac{15}{4}e^{2}-\frac{135}{16}e_{,}^{2}-\frac{5}{16}e_{,}^{2}+5\right.$$

$$\left.+\frac{15}{2}e^{2}+\frac{15}{2}e_{,}^{2}+\frac{155}{12}e_{,}^{2}-\frac{145}{16}e_{,}^{2}\right\}e_{,}^{2}\cos 2x$$

[17]

$$+\left\{\frac{1}{2}-\frac{5}{4}e_{,}^{2}-\frac{5}{4}e^{2}+\frac{5}{2}e_{,}^{2}+\frac{125}{96}e_{,}^{2}+\frac{5}{4}-\frac{95}{32}e_{,}^{2}-\frac{25}{8}e^{2}+\frac{135}{32}e_{,}^{2}+\frac{5}{2}-\frac{25}{4}e^{2}-\frac{25}{4}e_{,}^{2}\right.$$

$$\left.+\frac{155}{24}e_{,}^{2}-\frac{435}{32}e_{,}^{2}\right\}e_{,}^{2}\cos(2t-2x)$$

[18]

$$+\left\{\frac{5}{4}-\frac{25}{8}e^2+\frac{25}{4}e_1^2-\frac{15}{4}+\frac{65}{32}e_1^2+\frac{75}{8}e^2-\frac{405}{32}e_1^2-\frac{35}{96}e_1^2+\frac{5}{2}-\frac{25}{4}e^2-\frac{25}{4}e_1^2\right.$$

$$\left.+\frac{155}{24}e_1^2+\frac{145}{32}e_1^2\right\}e_1^2\cos(2t+2z)$$

[19]

$$-\frac{e^3}{8}\cos 3x-\frac{7}{48}e^3\cos(2t-3x)+\frac{25}{48}e^3\cos(2t+3x)+\left\{\frac{1}{4}-\frac{5}{8}\right\}e^2e_1\cos(2x+z)$$

[20] [21] [22] [23]

$$+\left\{\frac{5}{4}+\frac{25}{8}\right\}e^2e_1\cos(2t-2x-z)-\left\{-\frac{3}{2}+\frac{5}{4}\right\}e^2e_1\cos(2t+2x+z)$$

[24] [25]

$$+\left\{\frac{1}{4}\quad\frac{5}{8}\right\}e^2e_1\cos(2x-z)+\left\{-\frac{15}{4}+\frac{25}{8}\right\}e^2e_1\cos(2t-2x+z)$$

[26] [27]

$$+\left\{\frac{1}{2}+\frac{5}{4}\right\}e^2e_1\cos(2t+2x-z)+\left\{\frac{1}{4}+\frac{5}{2}-5\right\}ee_1^2\cos(x+2z)$$

[28] [29]

$$+\left\{-\frac{3}{2}-\frac{15}{4}-\frac{15}{2}\right\}ee_1^2\cos(2t-x-2z)$$

[30]

$$+\left\{\frac{5}{4}-\frac{15}{4}+\frac{5}{2}\right\}ee_1^2\cos(2t+x+2z)+\left\{\frac{1}{4}+\frac{5}{2}-5\right\}ee_1^2\cos(x-2z)$$

[31] [32]

$$+\left\{-\frac{15}{4}+\frac{45}{4}-\frac{15}{2}\right\}ee_1^2\cos(2t-x+2z)+\left\{\frac{1}{2}+\frac{5}{4}+\frac{5}{2}\right\}ee_1^2\cos(2t+x-2z)$$

[33] [34]

$$+\left\{-\frac{1}{8}-\frac{5}{8}-5+\frac{145}{16}\right\}e_1^3\cos 3z+\left\{\frac{25}{48}+\frac{5}{4}+\frac{5}{2}+\frac{145}{32}\right\}e_1^3\cos(2t-3z)$$

[35] [36]

$$+\left\{-\frac{7}{48}+\frac{25}{8}-\frac{15}{2}+\frac{145}{32}\right\}e_1^3\cos(2t+3z)$$

[37]

$$-\frac{e^4}{12}\cos 4x-\frac{e^4}{32}\cos(2t-4x)+\frac{9}{16}e^4\cos(2t+4x)$$

[38] [39] [40]

$$+\left\{\frac{1}{8}-\frac{5}{16}\right\}e^3e_1\cos(3x+z)+\left\{-\frac{7}{48}-\frac{35}{96}\right\}e^3e_1\cos(2t-3x-z)$$

[41] [42]

$$+\left\{-\frac{25}{16}+\frac{125}{96}\right\}e^3e_1\cos(2t-3x-z)+\left\{\frac{1}{8}-\frac{5}{16}\right\}e^3e_1\cos(3x-z)$$

[43] [44]

$$+\left\{\frac{7}{16}-\frac{35}{96}\right\}e^3 e_i \cos(2t-3x+z)$$
[45]

$$+\left\{\frac{25}{48}+\frac{125}{96}\right\}e^3 e_i \cos(2t+3x-z)+\left\{\frac{1}{16}+\frac{5}{8}-\frac{5}{4}\right\}e^2 e_i^2 \cos(2x+2z)$$
[46] [47]

$$+\left\{\frac{5}{4}+\frac{25}{8}+\frac{25}{4}\right\}e^2 e_i^2 \cos(2t-2x-2z)+\left\{\frac{5}{4}-\frac{15}{4}+\frac{5}{2}\right\}e^2 e_i^2 \cos(2t+2x+2z)$$
[48] [49]

$$+\left\{\frac{1}{16}+\frac{5}{8}-\frac{5}{4}\right\}e^2 e_i^2 \cos(2x-2z)+\left\{\frac{25}{8}-\frac{75}{8}+\frac{25}{4}\right\}e^2 e_i^2 \cos(2t-2x+2z)$$
[50] [51]

$$+\left\{\frac{1}{2}+\frac{5}{4}+\frac{5}{2}\right\}e^2 e_i^2 \cos(2t+2x-2z)+\left\{\frac{1}{8}+\frac{5}{8}+5-\frac{145}{16}\right\}e\, e_i^3 \cos(x+3z)$$
[52] [53]

$$+\left\{-\frac{25}{16}-\frac{15}{4}-\frac{15}{2}-\frac{435}{32}\right\}e\, e_i^3 \cos(2t-x-3z)$$
[54]

$$+\left\{-\frac{7}{48}+\frac{25}{8}-\frac{15}{2}+\frac{145}{32}\right\}e\, e_i^3 \cos(2t+x+3z)$$
[55]

$$+\left\{\frac{1}{8}+\frac{5}{8}+5-\frac{145}{16}\right\}e\, e_i^3 \cos(x-3z)+\left\{\frac{7}{16}-\frac{75}{8}+\frac{45}{2}-\frac{435}{32}\right\}e\, e_i^3 \cos(2t-x+3z)$$
[56] [57]

$$+\left\{\frac{25}{48}+\frac{5}{4}+\frac{5}{2}-\frac{145}{32}\right\}e\, e_i^3 \cos(2t+x-3z)-\left\{-\frac{1}{12}-\frac{5}{16}-\frac{5}{4}-\frac{145}{16}+\frac{745}{96}\right\}e_i^4 \cos 4z$$
[58] [59]

$$+\left\{\frac{9}{16}+\frac{125}{96}+\frac{5}{2}+\frac{145}{32}+\frac{745}{192}\right\}e_i^4 \cos(2t-4z)$$
[60]

$$+\left\{-\frac{1}{32}-\frac{35}{96}+\frac{25}{4}-\frac{435}{32}+\frac{745}{192}\right\}e_i^4 \cos(2t+4z)$$
[61]

Terms in $R$ multiplied by $-\frac{3}{2}\sin^2\frac{i}{2}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}$

$$=\left\{1-\frac{5}{2}e^2+\frac{3}{2}e_i^2+5e_i^2-\frac{5}{2}e_i^2-\frac{5}{2}e_i^2\right\}\cos 2y$$
[62]

$$+\left\{1+\frac{3}{2}e^2-\frac{5}{2}e_i^2+5e_i^2+\frac{5}{2}e_i^2-\frac{15}{2}e_i^2\right\}\cos(2t-2y)$$
[63]

$$-3e\cos(x-2y)+e\cos(x+2y)-e\cos(2t-x-2y)-e\cos(2t+x-2y)$$
[65] [66] [67] [69]

$$+\left\{-1+\frac{5}{2}\right\}e_1\cos(x-2y)+\left\{-1+\frac{5}{2}\right\}e_1\cos(x+2y)+\left\{1+\frac{5}{2}\right\}e_1\cos(2t-x-2y)$$
[71] [72] [73]

$$+\left\{-3+\frac{5}{2}\right\}e_1\cos(2t+x-2y)+\frac{5}{2}e^2\cos(2x-2y)+e^2\cos(2x+2y)$$
[75] [77] [78]

$$-\frac{e^2}{4}\cos(2t-2x-2y)-\frac{e^2}{4}\cos(2t+2x-2y)$$
[79] [81]

$$+\left\{3-\frac{15}{2}\right\}ee_1\cos(x+z-2y)+\left\{-1+\frac{5}{2}\right\}ee_1\cos(x+z+2y)$$
[83] [84]

$$+\left\{-1-\frac{5}{2}\right\}ee_1\cos(2t-x-z-2y)$$
[85]

$$+\left\{3-\frac{5}{2}\right\}ee_1\cos(2t+x+z-2y)+\left\{3-\frac{15}{2}\right\}ee_1\cos(2t-z-2y)$$
[87] [89]

$$+\left\{-1+\frac{5}{2}\right\}ee_1\cos(x-z+2y)+\left\{3-\frac{5}{2}\right\}ee_1\cos(2t-x+z-2y)$$
[90] [91]

$$+\left\{-1-\frac{5}{2}\right\}ee_1\cos(2t+x-z-2y)+\left\{-\frac{3}{8}-\frac{5}{2}+5\right\}e_1^2\cos(2z-2y)$$
[93] [95]

$$+\left\{-\frac{5}{8}-\frac{5}{2}+5\right\}e_1^2\cos(2z+2y)+\left\{\frac{5}{8}+\frac{5}{2}+5\right\}e_1^2\cos(2t-2z-2y)$$
[96] [97]

$$+\left\{\frac{19}{8}-\frac{15}{2}+5\right\}e_1^2\cos(2t+2z-2y)$$
[99]

## Terms in $R$ multiplied by $-\frac{3}{2}\cos^4\frac{i}{2}\frac{a^3}{a_1^3}$

$$=\frac{1}{2}+\frac{3}{4}e^2+\frac{3}{4}e_1^2+\frac{15}{16}e_1^4+\frac{9}{8}e^2e_1^2+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_1^2+\frac{23}{32}e^4+\frac{13}{32}e_1^4+\frac{25}{8}e^2e_1^2\right\}\cos 2t$$
[1]

$$+\left\{-1+\frac{e^2}{8}-\frac{3}{2}e_1^2\right\}e\cos x+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{15}{4}e_1^2\right\}e\cos(2t-x)$$
[2] [3]

$$+\left\{\frac{1}{2}-\frac{19}{16}e^2-\frac{5}{4}e_1^2\right\}e\cos(2t+x)+\left\{\frac{3}{2}+\frac{9}{4}e^2+\frac{27}{16}e_1^2\right\}e_1\cos z$$
[4] [5]

$$+\left\{\frac{7}{4}-\frac{35}{8}e^2-\frac{123}{32}e_1^2\right\}e_1\cos(2t-z)$$
[6]

$$+\left\{-\frac{1}{4}+\frac{5}{8}e^2+e_1^2\right\}e_1\cos(2t+z)+\left\{-\frac{1}{4}+\frac{e^2}{12}-\frac{3}{8}e_1^2\right\}e^2\cos 2x$$
[7] [8]

$$+\left\{\frac{5}{4}-\frac{25}{8}e_1^2\right\}e^2\cos(2t-2x)+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_1^2\right\}e^2\cos(2t+2x)$$
[9] [10]

$$+\left\{-\frac{3}{2}+\frac{3}{16}e^2-\frac{27}{16}e_1^2\right\}ee_1\cos(x+z)+\left\{-\frac{21}{4}+\frac{91}{32}e^2+\frac{369}{32}e_1^2\right\}ee_1\cos(2t-x-z)$$
[11] [12]

$$+\left\{-\frac{1}{4}+\frac{19}{32}e^2+\frac{e_1^2}{32}\right\}ee_1\cos(2t+x+z)+\left\{-\frac{3}{2}+\frac{3}{16}e^2-\frac{27}{16}e_1^2\right\}ee_1\cos(x-z)$$
[13] [14]

$$+\left\{\frac{3}{4}-\frac{13}{32}e^2-\frac{3}{32}e_1^2\right\}ee_1\cos(2t-x+z)+\left\{\frac{7}{4}-\frac{133}{32}e^2-\frac{123}{32}e_1^2\right\}ee_1\cos(2t+x-z)$$
[15] [16]

$$+\left\{\frac{9}{4}+\frac{27}{8}e^2+\frac{21}{12}e_1^2\right\}e_1^2\cos 2z+\left\{\frac{17}{4}-\frac{85}{8}e^2-\frac{115}{12}e_1^2\right\}e_1^2\cos(2t-2z)^*-\frac{e^3}{8}\cos 3x$$
[17] [18] [20]

$$-\frac{7}{48}e^3\cos(2t-3x)+\frac{25}{48}e^3\cos(2t+3x)-\frac{3}{8}e^2e_1\cos(2x+z)$$
[21] [22] [23]

$$+\frac{35}{8}e^2e_1\cos(2t-2x-z)-\frac{e^2e_1}{4}\cos(2t+2x+z)-\frac{3}{8}e^2e_1\cos(2x-z)$$
[24] [25] [26]

$$-\frac{5}{8}e^2e_1\cos(2t-2x+z)+\frac{7}{4}e^2e_1\cos(2t+2x-z)$$
[27] [28]

$$-\frac{9}{4}ee_1^2\cos(x+2z)-\frac{51}{4}ee_1^2\cos(2t-x-2z)$$
[29] [30]

$$-\frac{9}{4}ee_1^2\cos(x-2z)+\frac{17}{4}ee_1^2\cos(2t+x-2z)$$
[32] [34]

$$+\frac{53}{16}e_1^3\cos 3z+\frac{845}{96}e_1^3\cos(2t-3z)+\frac{e^2e_1}{96}\cos(2t+3z)-\frac{e^4}{12}\cos 4x-\frac{e^4}{32}\cos(2t-4x)$$
[35] [36] [37] [38] [39]

* It is remarkable that the coefficient of argument 19 equals zero.

$$+\frac{9}{16}e^4\cos(2t+4x)-\frac{3}{16}e^3e_i\cos(3x+z)$$
[40] [41]

$$-\frac{49}{96}e^3e_i\cos(2t+3x-z)-\frac{25}{96}e^3e_i\cos(2t-3x-z)-\frac{3}{16}e^3e_i\cos(3x-z)$$
[42] [43] [44]

$$+\frac{7}{96}e^3e_i\cos(2t-3x+z)+\frac{175}{96}e^3e_i\cos(2t+3x-z)-\frac{9}{16}e^3e_i^3\cos(2x+2z)$$
[45] [46] [47]

$$+\frac{85}{8}e^3e_i^3\cos(2t-2x-2z)-\frac{9}{16}e^3e_i^3\cos(2x-2z)+\frac{17}{4}e^3e_i^3\cos(2t+2x-2z)$$
[48] [50] [52]

$$-\frac{53}{16}ee_i^3\cos(x+3z)-\frac{845}{32}ee_i^3\cos(2t\quad x-3z)+\frac{ee_i^3}{96}\cos(2t+x-3z)$$
[53] [54] [55]

$$-\frac{53}{16}ee_i^3\cos(x-3z)-\frac{ee_i^3}{32}\cos(2t-x+3z)-\frac{25}{96}ee_i^3\cos(2t+x-3z)$$
[56] [57] [58]

$$-\frac{283}{96}e_i^4\cos 4z+\frac{2453}{192}e_i^4\cos(2t-4z)-\frac{741}{192}e_i^4\cos(2t+4z)$$
[59] [60] [61]

## Terms in $R$ multiplied by $-\frac{3}{2}\sin^2\frac{i}{2}\cos^2\frac{i}{2}\frac{a^3}{a_i^3}$ or $-\frac{3}{8}\sin^2 i\frac{a^3}{a_i^3}$

$$=\left\{1-\frac{5}{2}e^2+\frac{3}{2}e_i^2\right\}\cos 2y+\left\{1+\frac{3}{2}e^2-\frac{5}{2}e_i^2\right\}\cos(2t-2y)-3e\cos(x-2y)$$
[62] [63] [65]

$$+e\cos(x+2y)-e\cos(2t-x-2y)-e\cos(2t+x-2y)+\frac{3}{2}e_i\cos(z-2y)+\frac{3}{2}e_i\cos(z+2y)$$
[66] [67] [69] [71] [72]

$$+\frac{7}{2}e_i\cos(2t-z-2y)-\frac{e_i}{2}\cos(2t+z-2y)+\frac{5}{2}e^2\cos(2x-2y)+e^2\cos(2x+2y)$$
[73] [75] [77] [78]

$$-\frac{e^2}{4}\cos(2t-2x-2y)-\frac{e^2}{4}\cos(2t+2x-2y)-\frac{9}{2}ee_i\cos(x+z-2y)$$
[79] [81] [83]

$$+\frac{3}{2}ee_i\cos(x+z+2y)-\frac{7}{2}ee_i\cos(2t-x-z-2y)+\frac{ee_i}{2}\cos(2t+x+z-2y)$$
[84] [85] [87]

$$-\frac{9}{2}ee_i\cos(x-z-2y)+\frac{3}{2}ee_i\cos(x-z+2y)+\frac{ee_i}{2}\cos(2t-x+z-2y)$$
[89] [90] [91]

$$-\frac{7}{2}ee_i\cos(2t+x-z-2y)+\frac{17}{8}e_i^2\cos(2z-2y)+\frac{15}{8}e_i^2\cos(2z+2y)$$
[93] [95] [96]

$$+\frac{65}{8}e_i^2\cos(2t-2z-2y)-\frac{e_i^2}{8}\cos(2t+2z-2y)$$
[97] [99]

$$R=m_i\left\{-\frac{1}{r_i}-\frac{1}{4}\left\{1+\frac{3}{2}e^2+\frac{3}{2}e_i^2+\frac{9}{4}e^2e_i^2+\frac{15}{8}e_i^4-\frac{3}{2}\gamma^2-\frac{9}{4}\gamma^2e^2-\frac{9}{4}\gamma^2e_i^2+\frac{39}{8}\gamma^4\right\}\frac{a^2}{a_i^3}\right.$$
[0]

$$-\frac{3}{4}\left\{1-\frac{5}{2}e^2-\frac{5}{2}e_i^2+\frac{23}{16}e^4+\frac{25}{4}e^2e_i^2+\frac{13}{16}e_i^4\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}\cos 2t$$
[1]

$$+\frac{1}{2}\left\{1-\frac{e^2}{8}-\frac{3}{2}e_i^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_i^3}e\cos x$$
[2]

$$+\frac{9}{4}\left\{1-\frac{13}{24}e^2-\frac{5}{2}e_i^2\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}e\cos(2t-x)$$
[3]

$$-\frac{3}{4}\left\{1-\frac{19}{8}e^2-\frac{5}{2}e_i^2\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}e\cos(2t+x)$$
[4]

$$-\frac{3}{4}\left\{1+\frac{3}{2}e^2+\frac{9}{8}e_i^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_i^3}e_i\cos z$$
[5]

$$-\frac{21}{8}\left\{1-\frac{5}{2}e^2-\frac{123}{56}e_i^2\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}e_i\cos(2t-z)$$
[6]

$$+\frac{3}{8}\left\{1-\frac{5}{2}e^2-4e_i^2\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}e_i\cos(2t+z)$$
[7]

$$+\frac{1}{8}\left\{1-\frac{e^2}{3}+\frac{3}{2}e_i^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_i^3}e^2\cos 2x$$
[8]

$$-\frac{15}{8}\left\{1-\frac{5}{2}e_i^2\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}e^2\cos(2t-2x)$$
[9]

$$-\frac{3}{4}\left\{1-\frac{5}{2}e^2-\frac{5}{2}e_i^2\right\}\cos^2\frac{i}{2}\frac{a^2}{a_i^3}e^2\cos(2t+2x)$$
[10]

Development of $R$.

$$+\frac{3}{4}\left\{1-\frac{e^2}{8}+\frac{9}{8}e_1^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_1^3}ee_1\cos(x+z) \qquad [11]$$

$$+\frac{63}{8}\left\{1-\frac{91}{168}e^2-\frac{123}{56}e_1^2\right\}\cos^4\frac{i}{2}\frac{a^2}{a_1^3}ee_1\cos(2t-x-z) \qquad [12]$$

$$+\frac{3}{8}\left\{1-\frac{19}{8}e^2-\frac{e_1^2}{8}\right\}\cos^4\frac{i}{2}\frac{a^2}{a_1^3}ee_1\cos(2t+x+z) \qquad [13]$$

$$+\frac{3}{4}\left\{1-\frac{e^2}{8}+\frac{9}{8}e_1^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_1^3}ee_1\cos(x-z) \qquad [14]$$

$$-\frac{9}{8}\left\{1-\frac{13}{24}e^2-\frac{e_1^2}{8}\right\}\cos^4\frac{i}{2}\frac{a^2}{a_1^3}ee_1\cos(2t-x+z) \qquad [15]$$

$$-\frac{21}{8}\left\{1-\frac{19}{8}e^2-\frac{123}{56}e_1^2\right\}\cos^4\frac{i}{2}\frac{a^2}{a_1^3}ee_1\cos(2t+x-z) \qquad [16]$$

$$-\frac{9}{8}\left\{1+\frac{3}{2}e^2+\frac{7}{9}e_1^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_1^3}e_1^2\cos 2z \qquad [17]$$

$$-\frac{51}{8}\left\{1-\frac{5}{2}e^2-\frac{115}{51}e_1^2\right\}\cos^4\frac{i}{2}\frac{a^2}{a_1^3}e_1^2\cos(2t-2z) \qquad [18]$$

$$+\frac{1}{16}\frac{a^2}{a_1^3}e^2\cos 3x \qquad [20]$$

$$+\frac{7}{32}\frac{a^2}{a_1^3}e^2\cos(2t-3x) \qquad [21]$$

$$-\frac{25}{32}\frac{a^2}{a_1^3}e^2\cos(2t+3x) \qquad [22]$$

$$+\frac{3}{16}\frac{a^2}{a_1^3}e^2e_1\cos(2x+z) \qquad [23]$$

$$-\frac{105}{16}\frac{a^2}{a_1^3}e^2e_1\cos(2t-2x-z) \qquad [24]$$

$$+\frac{3}{8}\frac{a^2}{a_1^3}e^2e_1\cos(2t+2x+z) \qquad [25]$$

$$+\frac{3}{16}\frac{a^2}{a_1^3}e^2e_1\cos(2x-z) \qquad [26]$$

$$+\frac{15}{16}\frac{a^2}{a_1^3}e^2e_1\cos(2t-2x+z) \qquad [27]$$

$$-\frac{21}{8}\frac{a^2}{a_1^3}e^2e_1\cos(2t+2x-z) \qquad [28]$$

$$+\frac{9}{8}\frac{a^2}{a_1^3}ee_1^2\cos(x+2z) \qquad [29]$$

$$+\frac{153}{8}\frac{a^2}{a_1^3}ee_1^2\cos(2t-x-2z) \qquad [30]$$

$$+\frac{9}{8}\frac{a^2}{a_1^3}ee_1^2\cos(x-2z) \qquad [32]$$

$$-\frac{51}{8}\frac{a^2}{a_1^3}ee_1^2\cos(2t+x-2z) \qquad [34]$$

$$-\frac{53}{32}\frac{a^2}{a_1^3}e_1^3\cos 3z \qquad [35]$$

$$-\frac{845}{64}\frac{a^2}{a_1^3}e_1^3\cos(2t-3z) \qquad [36]$$

$$-\frac{1}{64}\frac{a^2}{a_1^3}e_1^3\cos(2t+3z) \qquad [37]$$

$$+\frac{1}{24}\frac{a^2}{a_1^3}e^4\cos 4x \qquad [38]$$

$$+\frac{3}{64}\frac{a^2}{a_1^3}e^4\cos(2t-4x) \qquad [39]$$

$$-\frac{27}{32}\frac{a^2}{a_1^3}e^4\cos(2t+4x) \qquad [40]$$

$$+\frac{3}{32}\frac{a^2}{a_1^3}e^3e_1\cos(3x+z) \qquad [41]$$

$$+\frac{49}{64}\frac{a^2}{a_j^3}e^3e_j\cos(2t-3x-z)+\frac{25}{64}\frac{a^2}{a_j^3}e^3e_j\cos(2t-3x-z)+\frac{8}{32}\frac{a^2}{a_j^3}e^3e_j\cos(3x-z)$$
[42] [44]

Development of $R$.

$$-\frac{7}{64}\frac{a^2}{a_j^3}e^3e_j\cos(2t-3x+z)-\frac{175}{64}\frac{a^2}{a_j^3}e^3e_j\cos(2t+3x-z)$$
[45] [46]

$$+\frac{9}{32}\frac{a^2}{a_j^3}e^3e_j^3\cos(2x+2z)-\frac{255}{16}e^3e_j^3\cos(2t-2x-2z)$$
[47] [48]

$$+\frac{9}{32}\frac{a^2}{a_j^3}e^3e_j^3\cos(2x-2z)-\frac{51}{8}\frac{a^2}{a_j^3}e^3e_j^3\cos(2t+2x-2z)+\frac{53}{32}\frac{a^2}{a_j^3}ee_j^3\cos(x+3z)$$
[50] [52] [53]

$$+\frac{2535}{64}\frac{a^2}{a_j^3}ee_j^3\cos(2t-x-3z)-\frac{1}{64}\frac{a^2}{a_j^3}ee_j^3\cos(2t+x+3z)$$
[54] [55]

$$+\frac{53}{32}\frac{a^2}{a_j^3}ee_j^3\cos(x-3z)+\frac{3}{64}\frac{a^2}{a_j^3}ee_j^3\cos(2t-x+3z)$$
[56] [57]

$$+\frac{45}{64}\frac{a^2}{a_j^3}ee_j^3\cos(2t+x-3z)+\frac{591}{64}\frac{a^2}{a_j^3}e_j^4\cos 4z$$
[58] [59]

$$-\frac{2453}{128}\frac{a^2}{a_j^3}e_j^4\cos(2t-4z)+\frac{741}{128}\frac{a^2}{a_j^3}e_j^4\cos(2t+4z)$$
[60] [61]

$$-\frac{3}{8}\left\{1-\frac{5}{2}e^2+\frac{3}{2}e_j^2\right\}\frac{a^2}{a_j^3}\gamma^2\cos 2y-\frac{3}{8}\left\{1+\frac{3}{2}e^2-\frac{5}{2}e_j^2+\frac{\gamma^2}{8}\right\}\frac{a^2}{a_j^3}\gamma^2\cos(2t-2y)$$
[62] [63]

$$+\frac{9}{8}\frac{a^2}{a_j^3}\gamma^2e\cos(x-2y)$$
[65]

$$-\frac{3}{8}\frac{a^2}{a_j^3}\gamma^2e\cos(x+2y)+\frac{3}{8}\frac{a^2}{a_j^3}\gamma^2e\cos(2t-x-2y)$$
[66] [67]

$$+\frac{3}{8}\frac{a^2}{a_j^3}\gamma^2e\cos(2t+x-2y)-\frac{9}{16}\frac{a^2}{a_j^3}\gamma^2e_j\cos(x-2y)$$
[69] [71]

$$-\frac{9}{16}\frac{a^2}{a_j^3}\gamma^2e_j\cos(x+2y)-\frac{21}{16}\frac{a^2}{a_j^3}\gamma^2e_j\cos(2t-x-2y)$$
[72] [73]

$$+\frac{3}{16}\frac{a^2}{a_j^3}\gamma^2e_j\cos(2t+x-2y)-\frac{15}{16}\frac{a^2}{a_j^3}\gamma^2e^2\cos(2x-2y)$$
[75] [77]

Development of $R$.

$$-\frac{3}{8}\frac{a^2}{a_i^3}\gamma^2 e^2\cos(2x+2y) \quad [78]$$

$$+\frac{3}{32}\frac{a^2}{a_i^3}\gamma^2 e^2\cos(2t-2x-2y) \quad [79]$$

$$+\frac{3}{32}\frac{a^2}{a_i^3}\gamma^2 e^2\cos(2t+2x-2y) \quad [81]$$

$$+\frac{27}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(x+z-2y) \quad [83]$$

$$-\frac{9}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(x+z+2y) \quad [84]$$

$$+\frac{21}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(2t-x-z-2y) \quad [85]$$

$$-\frac{3}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(2t+x+z-2y) \quad [87]$$

$$+\frac{27}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(x-z-2y) \quad [89]$$

$$-\frac{9}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(x-z+2y) \quad [90]$$

$$-\frac{3}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(2t-x+z-2y) \quad [91]$$

$$+\frac{21}{16}\frac{a^2}{a_i^3}\gamma^2 e e_i\cos(2t+x-z-2y) \quad [93]$$

$$-\frac{51}{64}\frac{a^2}{a_i^3}\gamma^2 e_i^2\cos(2z-2y) \quad [95]$$

$$-\frac{45}{64}\frac{a^2}{a_i^3}\gamma^2 e_i^2\cos(2z+2y) \quad [96]$$

$$-\frac{195}{64}\frac{a^2}{a_i^3}\gamma^2 e_i^2\cos(2t-2z-2y) \quad [97]$$

$$+\frac{3}{64}\frac{a^2}{a_i^3}\gamma^2 e_i^2\cos(2t+2z-2y) \quad [99]$$

$$-\frac{3}{8}\left\{1+3e^2+3e_i^2-\frac{11}{4}\gamma^2\right\}\frac{a^3}{a_i^4}{}^{*}\cos t \quad [101]$$

$$+\frac{15}{16}\frac{a^3}{a_i^4}e\cos(t-x) \quad [102]$$

$$+\frac{3}{16}\frac{a^3}{a_i^4}e\cos(t+x) \quad [103]$$

$$-\frac{9}{8}\frac{a^3}{a_i^4}e_i\cos(t-z) \quad [104]$$

$$-\frac{3}{8}\frac{a^3}{a_i^4}e_i\cos(t+x) \quad [105]$$

$$-\frac{33}{64}\frac{a^3}{a_i^4}e^2\cos(t-2x) \quad [106]$$

$$+\frac{9}{64}\frac{a^3}{a_i^4}e^2\cos(t+2x) \quad [107]$$

$$+\frac{45}{16}\frac{a^3}{a_i^4}e e_i\cos(t-x-z) \quad [108]$$

$$+\frac{3}{16}\frac{a^3}{a_i^4}e e_i\cos(t+x+z) \quad [109]$$

$$+\frac{15}{16}\frac{a^3}{a_i^4}e e_i\cos(t-x+z) \quad [110]$$

$$+\frac{9}{16}\frac{a^3}{a_i^4}e e_i\cos(t+x-z) \quad [111]$$

$$-\frac{159}{64}\frac{a^3}{a_i^4}e_i^2\cos(t-2z) \quad [112]$$

$$-\frac{33}{64}\frac{a^3}{a_i^4}e_i^2\cos(t+2z) \quad [113]$$

$$-\frac{9}{16}\frac{a^3}{a_i^4}\gamma^2\cos(t-2y) \quad [114]$$

$$-\frac{15}{32}\frac{a^3}{a_i^4}\sin^2\frac{i}{2}\cos(t+2y) \quad [115]$$

$$-\frac{5}{8}\left\{1-6e^2-6e_i^2-\frac{3}{4}\gamma^2\right\}\frac{a^3}{a_i^4}\cos 3t \quad [116]$$

* For the coefficients of the terms multiplied by $\frac{a^3}{a_i^4}$ see p. 39.

$$+\frac{45}{16}\frac{a^3}{a_i^4}e\cos(3t-x)-\frac{15}{16}\frac{a^3}{a_i^4}e\cos(3t+x)-\frac{25}{8}\frac{a^3}{a_i^4}e_i\cos(3t-z)$$
[117] [118] [119]

$$+\frac{5}{8}\frac{a^3}{a_i^4}e_i\cos(3t+z)-\frac{285}{64}\frac{a^3}{a_i^4}e^2\cos(3t-2x)-\frac{75}{64}\frac{a^3}{a_i^4}e^2\cos(3t+2x)$$
[120] [121] [122]

$$-\frac{225}{16}\frac{a^3}{a_i^4}ee_i\cos(3t-x-z)+\frac{15}{16}\frac{a^3}{a_i^4}ee_i\cos(3t+x+z)$$
[123] [124]

$$-\frac{45}{16}\frac{a^3}{a_i^4}ee_i\cos(3t-x+z)-\frac{75}{16}\frac{a^3}{a_i^4}ee_i\cos(3t+x-z)$$
[125] [126]

$$-\frac{635}{64}\frac{a^3}{a_i^4}e_i^2\cos(3t-2z)-\frac{5}{64}\frac{a^3}{a_i^4}e_i^2\cos(3t+2z)-\frac{15}{32}\frac{a^3}{a_i^4}\gamma^2\cos(3t-2y)$$
[127] [128] [129]

In the elliptic movement ;

$$s=\gamma\sin(g\lambda-\nu)$$

$$\lambda=nt+2e\sin x+\frac{5}{4}e^2\sin 2x$$

$$s=\gamma(1-e^2)\sin y+\gamma e\sin(x-y)+\gamma e\sin(x+y)+\gamma\frac{e^2}{8}\sin(2x-y)+\frac{9}{8}\gamma e^2\sin(2x+y)$$
[146] [149] [150] [161] [162]

$$s^2=\frac{\gamma^2}{2}-\frac{\gamma^2}{2}(1-4e^2)\cos 2y+\gamma^2 e\cos(x-2y)-\gamma^2 e\cos(x+2y)$$
[62] [65] [66]

$$+\frac{5}{8}\gamma^2e^2\cos(2x-2y)-\frac{5}{8}\gamma^2e^2\cos(2x+2y)$$
[77] [78]

$$z^*=a\gamma\left(1-\frac{e^2}{2}\right)\sin y+\frac{3a\gamma e}{2}\sin(x-y)+\frac{a\gamma e}{2}\sin(x+y)$$
[146] [149] [150]

$$-\frac{a\gamma e^2}{8}\sin(2x-y)+\frac{3a\gamma e^2}{8}\sin(2x+y)$$
[161] [162]

$$\frac{s}{r}=\frac{\gamma}{a}(1-e^2)\sin y+\frac{\gamma e}{2a}\sin(x-y)+\frac{3\gamma e}{2a}\sin(x+y)$$
[146] [149] [150]

* This quantity $z$, which is one of the rectangular coordinates of the moon, must not be confounded with $z=n_i t-\varpi_i$; this latter quantity should rather be $x_i$, but I think it better to conform as far as possible to the notation of M. Damoiseau.

$$+\frac{\gamma}{8}\frac{e^5}{a}\sin(2x-y)+\frac{17}{8}\frac{\gamma e^3}{a}\sin(2x+y)$$
[161] [162]

$$\frac{q}{r}\delta\cdot\frac{1}{r}=\left\{(1-e^2)r_0+\frac{e^2}{2}r_2\right\}\frac{\gamma}{a^2}\sin y-\left\{(1-e^2)\frac{r_1}{2}+\frac{e^2}{4}r_3-\frac{3e^2}{4}r_4\right\}\frac{\gamma}{a^2}\sin(2t-y)$$
[146] [147]

$$+\left\{(1-e^2)\frac{r_1}{2}-\frac{e^2}{4}r_4+\frac{3e^2}{4}r_5\right\}\frac{e\gamma}{a^2}\sin(2t+y)+\frac{e\gamma r_0}{2a^2}\sin(x-y)$$
[148] [149]

$$+\frac{3r_0}{2a^2}e\gamma\sin(x+y)+\left\{-\frac{r_2}{2}-\frac{3r_3}{4}\right\}\frac{e\gamma}{a^2}\sin(2t-x-y)$$
[150] [151]

$$+\left\{\frac{r_3}{2}-\frac{r_1}{4}\right\}\frac{e\gamma}{a^2}\sin(2t-x+y)+\left\{-\frac{r_4}{2}-\frac{r_1}{4}\right\}\frac{e\gamma}{a^2}\sin(2t+x-y)$$
[152] [153]

$$+\left\{\frac{r_4}{2}+\frac{3}{4}r_1\right\}\frac{e\gamma}{a^2}\sin(2t+x+y)+\frac{r_2e_1\gamma}{2a^2}\sin(z-y)+\frac{r_2e_1\gamma}{2a^2}\sin(z+y)$$
[154] [155] [156]

$$-\frac{r_6e_1\gamma}{2a^2}\sin(2t-z-y)+\frac{r_6e_1\gamma}{2a^2}\sin(2t-z+y)-\frac{r_7e_1\gamma}{2a^2}\sin(2t+z-y)$$
[157] [158] [159]

$$+\left\{-\frac{r_0}{2}-\frac{3}{4}r_3-\frac{17}{16}r_1\right\}\frac{e^3\gamma}{a^2}\sin(2t-2x-y)+\left\{\frac{r_9}{2}-\frac{r_3}{4}-\frac{r_1}{16}\right\}\frac{e^3\gamma}{a^2}\sin(2t-2x+y)$$
[163] [164]

$$+\left\{-\frac{r_{10}}{2}+\frac{r_4}{4}+\frac{r_1}{16}\right\}\frac{e^3\gamma}{a^2}\sin(2t+2x-y)+\left\{\frac{r_{10}}{2}+\frac{3r_4}{4}+\frac{17}{16}r_1\right\}\frac{e^3\gamma}{a^2}\sin(2t+2x+y)$$
[165] [166]

$$+\left\{-\frac{r_{11}}{2}+\frac{r_5}{4}\right\}\frac{ee_1\gamma}{a^2}\sin(x+z-y)+\left\{\frac{r_{11}}{2}+\frac{3r_5}{4}\right\}\frac{ee_1\gamma}{a^2}\sin(x+z+y)$$
[167] [168]

$$+\left\{-\frac{r_{12}}{2}+\frac{r_6}{4}-\frac{3}{4}r_6\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x-z-y)+\left\{\frac{r_{12}}{2}+\frac{r_6}{4}\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x-z+y)$$
[169] [170]

$$+\left\{-\frac{r_{13}}{2}+\frac{r_7}{4}\right\}\frac{ee_1\gamma}{a^2}\sin(2t+x+z-y)+\left\{\frac{r_{13}}{2}+\frac{3}{4}r_7\right\}\frac{ee_1\gamma}{a^2}\sin(2t+x+z+y)$$
(171) (172)

$$+\left\{-\frac{r_{14}}{2}+\frac{r_8}{2}\right\}\frac{ee_1\gamma}{a^2}\sin(x-z-y)+\left\{\frac{r_{14}}{2}+\frac{3}{4}r_8\right\}\frac{ee_1\gamma}{a^2}\sin(x-z+y)$$
(173) (174)

$$+\left\{-\frac{r_{15}}{2}-\frac{3}{4}r_7\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x+z-y)+\left\{\frac{r_{15}}{2}-\frac{r_7}{2}\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x+z+y)$$
[175] [176]

$$-\frac{r_{16}}{2}\frac{e\,e_i\gamma}{a^2}\sin(2t+x-z-y)+\left\{\frac{r_{16}}{2}+\frac{3}{4}r_6\right\}\frac{e\,e_i\gamma}{a^2}\sin(2t+x-z+y)$$
[177] [178]

$$-\frac{r_{17}}{2}\frac{e_i^2\gamma}{a^2}\sin(2z-y)+\frac{r_{17}}{2}\frac{e_i^2\gamma}{a^2}\sin(2z+y)-\frac{r_{18}}{2}\frac{e_i^2\gamma}{a^2}\sin(2t-2z-y)$$
[179] [180] [181]

$$+\frac{r_{18}}{2}\frac{e_i^2\gamma}{a^2}\sin(2t-2z+y)-\frac{r_{19}}{2}\frac{e_i^2\gamma}{a^2}\sin(2t+2z-y)+\frac{r_{19}}{2}\frac{e_i^2\gamma}{a^2}\sin(2t+2z+y)$$
[182] [183] [184]

$$\frac{m_i z}{r^3}=\frac{m_i a\gamma}{a_i^3}\left(1+\frac{3}{2}e_i^2-\frac{e^2}{2}\right)\sin y+\frac{3m_i a\gamma e}{2a_i^3}\sin(x-y)+\frac{m_i a\gamma}{2a_i^3}\sin(x+y)$$
[146] [149] [150]

$$-\frac{3m_i a\gamma e_i}{2a_i^3}\sin(z-y)+\frac{3m_i a\gamma e_i}{2a_i^3}\sin(z+y)-\frac{m_i a\gamma e^2}{8a_i^3}\sin(2z-y)$$
[155] [156] [161]

$$+\frac{3m_i a\gamma e^2}{8a_i^3}\sin(2z+y)+\frac{9m_i a\gamma e\,e_i}{4a_i^3}\sin(x+z-y)+\frac{3m_i a\gamma e\,e_i}{4a_i^3}\sin(x+z+y)$$
[162] [167] [168]

$$+\frac{9m_i a\gamma e\,e_i}{4a_i^3}\sin(x-z-y)+\frac{3m_i a\gamma e\,e_i}{4a_i^3}\sin(x-z+y)-\frac{9m_i a\gamma e_i^2}{4a_i^3}\sin(2z-y)$$
[173] [174] [179]

$$+\frac{9m_i a\gamma e_i^2}{4a_i^3}\sin(2z+y)$$
[180]

$$\frac{a^3}{r^3}=1+\frac{3}{2}e^2+3e\cos x+\frac{9}{2}e^2\cos 2x$$

r being the elliptic value of $r$.

If $z=a\gamma z_{146}\sin y+a\gamma z_{147}\sin(2t-y)+a\gamma z_{148}\sin(2t+y)$ &c.

$$\frac{z}{r^3}^{*}=\left\{\left(1+\frac{3e^2}{2}\right)z_{146}+\frac{3}{2}e^2z_{150}-\frac{3e^2}{2}z_{149}\right\}\frac{\gamma}{a^2}\sin y$$
[146]

$$+\left\{\left(1+\frac{3e^2}{2}\right)z_{147}+\frac{3e^2}{2}z_{151}+\frac{3e^2}{2}z_{153}\right\}\frac{\gamma}{a^2}\sin(2t-y)$$
[147]

$$+\left\{\left(1+\frac{3e^2}{2}\right)z_{148}+\frac{3e^2}{2}z_{152}+\frac{3e^2}{2}z_{154}\right\}\frac{\gamma}{a^2}\sin(2t+y)$$
[148]

* This multiplication of $z$ by $r^{-3}$ may be effected at once by means of Table II.

$$+\left\{z_{149}-\frac{3}{2}z_{146}\right\}\frac{e\gamma}{a^2}\sin(x-y)+\left\{z_{150}+\frac{3}{2}z_{146}\right\}\frac{e\gamma}{a^2}\sin(x+y)$$
[149] [150]

$$+\left\{z_{151}+\frac{3}{2}z_{147}\right\}\frac{e\gamma}{a^2}\sin(2t-x-y)+\left\{z_{152}+\frac{3}{2}z_{148}\right\}\frac{e\gamma}{a^2}\sin(2t-x+y)$$
[151] [152]

$$+\left\{z_{153}+\frac{3}{2}z_{147}\right\}\frac{e\gamma}{a^2}\sin(2t+x-y)$$
[153]

$$+\left\{z_{154}+\frac{3}{2}z_{149}\right\}\frac{e\gamma}{a^2}\sin(2t+x+y)+z_{155}\frac{e\gamma}{a^2}\sin(x-y)+z_{156}\frac{e\gamma}{a^2}\sin(x+y)$$
[154] [155] [156]

$$+\left\{z_{161}+\frac{3}{2}z_{149}-\frac{9}{4}z_{146}\right\}\frac{e^2\gamma}{a^2}\sin(2x-y)+\left\{z_{162}+\frac{3}{2}z_{150}+\frac{9}{4}z_{146}\right\}\frac{e^2\gamma}{a^2}\sin(2x+y)$$
[161] [162]

$$+\left\{z_{163}+\frac{3}{2}z_{151}+\frac{9}{4}z_{147}\right\}\frac{e^2\gamma}{a^2}\sin(2t-2x-y)$$
[163]

$$+\left\{z_{164}+\frac{3}{2}z_{152}+\frac{9}{4}z_{148}\right\}\frac{e^2\gamma}{a^2}\sin(2t-2x+y)$$
[164]

$$+\left\{z_{165}+\frac{3}{2}z_{153}+\frac{9}{4}z_{147}\right\}\frac{e^2\gamma}{a^2}\sin(2t+2x-y)$$
[165]

$$+\left\{z_{166}+\frac{3}{2}z_{154}+\frac{9}{4}z_{148}\right\}\frac{e^2\gamma}{a^2}\sin(2t+2x+y)+\&c.$$
[166]

$$+\left\{z_{167}+\frac{3}{2}z_{155}\right\}\frac{ee_1\gamma}{a^2}\sin(x+z-y)+\left\{z_{168}+\frac{3}{2}z_{156}\right\}\frac{ee_1\gamma}{a^2}\sin(x+z+y)$$
[167] [168]

$$+\left\{z_{169}+\frac{3}{2}z_{157}\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x-z-y)+\left\{z_{170}+\frac{3}{2}z_{158}\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x-z+y)$$
[169] [170]

$$+\left\{z_{171}+\frac{3}{2}z_{159}\right\}\frac{ee_1\gamma}{a^2}\sin(2t+x+z-y)+\left\{z_{172}+\frac{3}{2}z_{160}\right\}\frac{ee_1\gamma}{a^2}\sin(2t+x+z+y)$$
[171] [172]

$$+\left\{z_{173}-\frac{3}{2}z_{156}\right\}\frac{ee_1\gamma}{a^2}\sin(x-z-y)+\left\{z_{174}-\frac{3}{2}z_{155}\right\}\frac{ee_1\gamma}{a^2}\sin(x-z+y)$$
[173] [174]

$$+\left\{z_{175}+\frac{3}{2}z_{159}\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x+z-y)+\left\{z_{176}+\frac{3}{2}z_{160}\right\}\frac{ee_1\gamma}{a^2}\sin(2t-x+z+y)$$
[175] [176]

$$+\left\{z_{177}+\frac{3}{2}z_{157}\right\}\frac{ee_1\gamma}{a^2}\sin(2t+x-z-y)+\left\{z_{178}+\frac{3}{2}z_{158}\right\}\frac{ee_1\gamma}{a^2}\sin(2t+x-z+y)$$
[177] [178]

$s = \frac{z}{r}$ nearly,

$$= \left\{ z_{146} + \frac{e^2}{2} z_{150} + \frac{e^2}{2} z_{140} \right\} \gamma \sin y$$

$$+ \left\{ z_{147} + \frac{e^2}{2} z_{151} + \frac{e^2}{2} z_{153} \right\} \gamma \sin (2t - y)$$

$$+ \left\{ z_{148} + \frac{e^2}{2} z_{152} + \frac{e^2}{2} z_{154} \right\} \gamma \sin (2t + y) + \&c.$$

$$\frac{d^2 . r^2}{2 . dt^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

$$\frac{d^2 z}{dt^2} + \frac{\mu z}{r^3} + \frac{m_i z}{\{r^2 - 2rr'\cos(\lambda - \lambda') + r_i^2\}^{\frac{3}{2}}}$$

$$r'^4 . \frac{d\lambda'^2}{dt^2} = h^2 - 2\int r'^2 \left(\frac{dR}{d\lambda'}\right) d\lambda'$$

Neglecting the square of the disturbing force

$$\frac{- d^2 . r^3 \delta . \frac{1}{r}}{dt^2} - \mu\delta . \frac{1}{r} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

$$\frac{d^2 z}{dt^2} + \frac{\mu z}{r^3} + \frac{m_i z}{r_i^3} + \frac{3 m_i z r' \cos(\lambda' - \lambda)}{r_i^3} = 0$$

$$\frac{d^2 . \delta z}{dt^2} + \frac{3\mu s\delta . \frac{1}{r}}{r} + \frac{\mu\delta . z}{r^3} + \frac{m_i z}{r_i^3} + \frac{3\mu_i z r' r \cos(\lambda' - \lambda)}{r_i^3} = 0$$

$$\frac{d\lambda'}{dt} = \frac{h(1+s^2)}{r^2} - \frac{(1+s^2)}{r^2}\int\left(\frac{dR}{d\lambda'}\right) dt$$

$$r\left(\frac{dR}{dr}\right) = a\left(\frac{dR}{da}\right), \quad \frac{dR}{d\lambda'} = \frac{dR}{dt}, \quad (t \text{ being used for } n\,t - n_i\,t).$$

Integrating the equation of p. 270, line 9, by the method of indeterminate coefficients, neglecting the cubes and higher powers of $e$ in order to obtain a first approximation, $m$ being equal to $\frac{n_i}{n}$ as in the notation of M. DAMOISEAU;

$$- r_0 - \frac{m_i a^3}{2\mu a_i^3}\left\{1 + \frac{3}{2}e^2 + \frac{3}{2}e_i^2 - \frac{3}{2}\gamma^2\right\} = 0$$

$$4(1-m)^2\left\{(1+3e^2)r_1 - \frac{3e^2}{2}\{r_3 + r_4\}\right\} - r_1$$

$$-\frac{3\,m_j\,a^3}{2\,\mu\,a_j^3}\left\{1-\frac{5}{2}\,e^2-\frac{5}{2}\,e_j^2-\frac{\gamma^2}{2}\right\}\left\{\frac{1}{1-m}+1\right\}=0$$

$$c^2\,*\left\{1-3\,r_0\right\}-1+\frac{2\,m_j\,a^3}{\mu\,a^3}=0$$

$$(2-2\,m-c)^2\left\{r_3-\frac{3}{2}\,r_1\right\}-r_3+\frac{9}{2}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2-c}{2-2\,m-c}+1\right\}=0$$

$$(2-2\,m+c)^2\left\{r_4-\frac{3}{2}\,r_1\right\}-r_4-\frac{3}{2}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2+c}{2-2\,m+c}+1\right\}=0$$

$$m^2\,r_5-r_5-\frac{3}{2}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}=0$$

$$(2-3\,m)^2\,r_6-r_6-\frac{21}{4}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2}{2-3\,m}+1\right\}=0$$

$$(2-m)^2\,r_7-r_7+\frac{3}{4}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2}{2-m}+1\right\}=0$$

$$4\,c^2\left\{(1-3\,r_0)\,r_8-\frac{3}{4}+3\,r_0\right\}-r_8+\frac{m_j}{2\,\mu}\,\frac{a^3}{a_j^3}=0$$

$$(2-2\,m-2\,c)^2\left\{r_9-\frac{3}{2}\,r_3\right\}-r_9-\frac{15}{4}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2-2\,c}{2-2\,m-2\,c}+1\right\}=0$$

$$(2-2\,m+2\,c)^2\left\{r_{10}-\frac{3}{2}\,r_4\right\}-r_{10}-\frac{3}{2}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2+2\,c}{2-2\,m+2\,c}+1\right\}=0$$

$$(c+m)^2\left\{r_{11}-\frac{3}{2}\,r_5\right\}-r_{11}+\frac{3}{2}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{c}{c+m}+1\right\}=0$$

$$(2-3\,m-c)^2\left\{r_{12}-\frac{3}{2}\,r_6\right\}-r_{12}+\frac{63}{4}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{c}{2-3\,m-c}+1\right\}=0$$

$$(2-m+c)^2\left\{r_{13}-\frac{3}{2}\,r_7\right\}-r_{13}+\frac{3}{4}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2+c}{2-m+c}+1\right\}=0$$

$$(c-m)^2\left\{r_{14}-\frac{3}{2}\,r_5\right\}-r_{14}+\frac{3}{2}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{c}{c-m}+1\right\}=0$$

$$(2-m-c)^2\left\{r_{15}-\frac{3}{2}\,r_7\right\}-r_{15}-\frac{9}{4}\,\frac{m_j}{\mu}\,\frac{a^3}{a_j^3}\left\{\frac{2-c}{2-m-c}+1\right\}=0$$

* The letter $c$ does not strictly denote the same quantity as in the notation of M. DAMOISEAU, or in that of the Mathematical Tracts, p. 33.

$$(2-3m+c)^2\left\{r_{16}-\frac{3}{2}r_6\right\}-r_{18}-\frac{21}{4}\frac{m_j}{\mu}\frac{a^3}{a_j^3}\left\{\frac{2+c}{2-3m+c}+1\right\}=0$$

$$4m^2r_{17}-r_{17}-\frac{9}{4}\frac{m_j}{\mu}\frac{a^3}{a_j^3}=0$$

$$(2-4m)^2r_{18}-r_{18}-\frac{51}{4}\frac{m_j}{\mu}\frac{a^3}{a_j^3}\left\{\frac{2}{2-4m}+1\right\}$$

$$4r_{19}-r_{19}=0$$

The equation for determining $z$ may be integrated in the same way.

$$-g^2z_{146}+3r_0+z_{146}+\frac{m_j}{\mu}\frac{a^3}{a_j^3}=0$$

$$-\left\{2(1-m)-g\right\}^2z_{147}-\frac{3r_1}{2}+z_{147}=0$$

$$-\left\{2(1-m)+g\right\}^2z_{148}+\frac{3r_1}{2}+z_{148}=0$$

$$-\left\{c-g\right\}^2z_{149}+\frac{3}{2}r_6+z_{149}-\frac{3}{2}z_{146}+\frac{3m_j}{2\mu}\frac{a^3}{a_j^3}=0$$

$$-\left\{c+g\right\}^2z_{150}+\frac{9}{2}r_6+z_{150}+\frac{3}{2}z_{146}+\frac{m_j}{\mu}\frac{a^3}{a_j^3}=0$$

$$-\left\{2(1-m)-c-g\right\}^2z_{147}+3\left\{-\frac{3r_1}{4}-\frac{r_3}{2}\right\}+z_{151}+\frac{3}{2}z_{147}=0$$

$$-\left\{2(1-m)-c+g\right\}^2z_{148}+3\left\{-\frac{r_1}{4}+\frac{r_3}{2}\right\}+z_{152}+\frac{3}{2}z_{148}=0$$

$$-\left\{2(1-m)+c-g\right\}^2z_{149}+3\left\{\frac{r_1}{4}-\frac{r_4}{2}\right\}+z_{153}+\frac{3}{2}z_{147}=0$$

$$-\left\{2(1-m)+c+g\right\}^2z_{150}+3\left\{\frac{3}{4}r_1+\frac{r_3}{2}\right\}+z_{154}+\frac{3}{2}z_{148}=0$$

$$-\left\{m-g\right\}^2z_{151}+\frac{3}{2}r_5+z_{155}-\frac{3m_j}{2\mu}\frac{a^3}{a_j^3}=0$$

$$-\left\{m+g\right\}^2z_{152}+\frac{3}{2}r_5+z_{156}+\frac{3m_j}{2\mu}\frac{a^3}{a_j^3}=0$$

$$-\left\{2(1-m)-m-g\right\}^2z_{153}-\frac{3}{2}r_6+z_{157}=0$$

$$-\left\{2(1-m)-m+g\right\}^2z_{154}+\frac{3}{2}r_6+z_{158}=0$$

$$-\left\{2(1-m)+m-g\right\}^2z_{155}-\frac{3}{2}r_7+z_{159}=0$$

$$-\left\{2(1-m)+m+g\right\}^{2} z_{156}+\frac{3}{2} r_7+z_{160}=0$$

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t}=\frac{h}{r^2}+\frac{2h}{r}\delta\cdot\frac{1}{r}+\frac{hz^2}{r^4}-\frac{(1+z^2)}{r^2}\int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t$$

$$\lambda'=\frac{h}{a^2}\left\{1+\frac{e^2}{2}+\frac{\gamma^2}{2}+2r_0\right\}t+\frac{2e(1+r_0)}{c}\sin x+\frac{5e^2(1+r_0)}{4c}\sin 2x$$

$$+\left\{2r_1+e^2(r_3+r_4)-\left\{-(1-\frac{5}{2}e^2-\frac{5}{2}e_i^2-\frac{\gamma^2}{2})\frac{3}{4(1-m)}+\frac{9e^2}{2(2-2m-c)}\right.\right.$$

$$\left.\left.-\frac{3e^2}{2(2-2m+c)}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{1}{2(1-m)}\sin 2t$$

$$+\left\{2r_3+e^2r_1-\left\{\frac{9}{2(2-m-c)}-\frac{3}{2(2-m)}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{e}{(2-2m-c)}\sin(2t-x)$$

$$+\left\{2r_4+e^2r_1-\left\{-\frac{3}{2(2-m+c)}-\frac{3}{2(2-m)}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{e}{(2-m+c)}\sin(2t+x)$$

$$+\frac{2r_5}{m}\sin z$$

$$+\left\{2r_6+\frac{21}{4(2-3m)}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{e_i}{(2-3m)}\sin(2t-z)$$

$$+\left\{2r_7-\frac{3}{4(2-m)}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{e_i}{(2-m)}\sin(2t+z)$$

$$+\left\{2r_9+r_3-\left\{-\frac{15}{4(2-2m-2c}+\frac{9}{2(2-2m-c}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{e^2}{2(1-m-c)}\sin(2t-2x)$$

$$+\left\{2r_{10}+r_4-\left\{-\frac{3}{2(2-2m+2c)}-\frac{3}{2(2-m+c)}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{e^2}{2(1-m+c)}\sin(2t+2x)$$

$$+\left\{2r_{11}+r_5\right\}\frac{ee_i}{(c+m)}\sin(x+z)$$

$$+\left\{2r_{12}+r_6-\left\{\frac{63}{4(2-3m-c)}-\frac{21}{4(2-3m)}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{ee_i}{(2-3m-c)}\sin(2t-x-z)$$

$$+\left\{2r_{13}+r_7-\left\{\frac{3}{4(2-m+c)}+\frac{3}{4(2-m)}\right\}\frac{m_i}{\mu}\frac{a^3}{a_i^3}\right\}\frac{ee_i}{(2-m+c)}\sin(2t+x+z)$$

Considering the terms which depend on the square of the disturbing force

$$\frac{\mathrm{d}^2\cdot r^4}{2\,\mathrm{d}t^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}R+r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=0$$

$$\frac{\mathrm{d}^2 . r^2}{2\,\mathrm{d}t^2} - \frac{\mathrm{d}^2 . r^2 \delta . \frac{1}{r}}{\mathrm{d}t^2} + \frac{3\,\mathrm{d}^2 . r^2 \left(\delta . \frac{1}{r}\right)^2}{2\,\mathrm{d}t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) = 0$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mu x}{r^3} + \frac{m_1 x}{r_1^3} - \frac{3\,m_1 x\, r\, r \cos(\lambda' - \lambda)}{r_1^5} = 0.$$

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h}{r^2}\left\{1 - \frac{1}{h}\int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t\left\{1 - \frac{1}{h^2}\int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t\right\} - \frac{1}{2h^2}\left\{\int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t\right\}^2\right.$$

$$= \frac{h(1+s^2)}{r^2} - \frac{(1+s^2)}{r^2}\int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t + \frac{(1+s^2)}{2r^2 h}\left\{\int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t\right\}^2$$

$\mathrm{d}R$ = the differential of $R$, supposing $n\,t$ only variable + the differential of $R$, with regard to $n_1 t$ only in as much as it is contained in the terms in $r$, $\lambda$ and $s$ due to the perturbations; hence

$\mathrm{d}R$ = the differential of $R$, supposing only $n\,t$ variable $+ \frac{\mathrm{d}R}{\mathrm{d}r}\,\mathrm{d}.\delta r' + \frac{\mathrm{d}R}{\mathrm{d}r}\,\mathrm{d}.\delta\lambda' + \frac{\mathrm{d}R}{\mathrm{d}r}\,\mathrm{d}.\delta s$; $\mathrm{d}.\delta r'$, $\mathrm{d}.\delta\lambda$, and $\mathrm{d}.\delta s$, being restrained to mean the differentials of those quantities with regard to $n_1 t$ only.

$$\delta R = \left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right)\delta r' + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\delta\lambda' + \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\delta s = -a\left(\frac{\mathrm{d}R}{\mathrm{d}a}\right)r'\delta . \frac{1}{r'} + \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\delta\lambda' + \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\delta s,$$

($t$ being used in the sense $n\,t - n_1 t$.) $\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\delta s = \frac{r^2}{2r_1^3}\, s\,\delta s$ nearly.

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}.\delta r + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}.\delta\lambda' + \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\mathrm{d}.\delta s = -a\left(\frac{\mathrm{d}R}{\mathrm{d}a}\right)\mathrm{d}.r\delta . \frac{1}{r} + \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)\mathrm{d}.\delta\lambda' + \left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\mathrm{d}.\delta s$$

$\mathrm{d}.r'\delta\frac{1}{r'}$, $\mathrm{d}.\delta\lambda'$ and $\mathrm{d}.\delta s$ being restrained to mean the differentials of those quantities with regard to $n_1 t$ only.

$$\delta . r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \mathrm{d}.\frac{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)}{\mathrm{d}r'}\,\delta r' + \mathrm{d}.\frac{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)}{\mathrm{d}\lambda'}\,\delta\lambda' + \mathrm{d}.\frac{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)}{\mathrm{d}s}\,\delta s$$

$$= -a\,\mathrm{d}.\frac{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)}{\mathrm{d}a}\,r'\delta . \frac{1}{r'} + \mathrm{d}.\frac{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)}{\mathrm{d}t}\,\delta\lambda' + \mathrm{d}.\frac{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)}{\mathrm{d}s}\,\delta s$$

$$\delta . \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right) = -a\,\mathrm{d}.\frac{\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)}{\mathrm{d}a}\,r'\delta . \frac{1}{r'} + \mathrm{d}.\frac{\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)}{\mathrm{d}t}\,\delta\lambda' + \frac{\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)}{\mathrm{d}s}\,\delta s$$

A similar theorem exists with the quantity $\delta . \frac{dR}{dz}$, and it will readily be seen that all the developments $\delta R$, $\delta . r \left(\frac{dR}{dr}\right)$, $\delta . \left(\frac{dR}{d\lambda}\right)$ and $\delta . \left(\frac{dR}{dz}\right)$ may be effected very easily by means of Table II.

Similarly, if $\delta'$ denote the variation due to the disturbance of the earth by the moon,

$$\delta' R = -a_1 d . \left(\frac{dR}{da_1}\right) r_1 \delta . \frac{1}{r_1} - d . \left(\frac{dR}{dt}\right) \delta\lambda_1$$

In $d R$ the terms which arise from

$$-a \left(\frac{dR}{da}\right) d . r \delta . \frac{1}{r} + \left(\frac{dR}{dt}\right) d . \delta\lambda + \left(\frac{dR}{dz}\right) d . \delta z$$

are multiplied by the small quantity $m$.

Considering in $r^2 \left(\frac{dR}{dr}\right)$ and $R$ the terms multiplied by $\frac{a^2}{a_1^2}$,

$$r^2 \left(\frac{dR}{dr}\right) = 2R, \qquad \delta . r^2 \left(\frac{dR}{dr}\right) = 2\delta R;$$

considering the terms multiplied by $\frac{a^3}{a_1^3}$,

$$r^3 \left(\frac{dR}{dr}\right) = 3R, \qquad \delta . r^3 \left(\frac{dR}{dr}\right) = 3\delta R$$

Hence the value of $r^3 \left(\frac{dR}{dr}\right)$ and $\delta . r^3 \left(\frac{dR}{dr}\right)$ may at once be inferred from $R$ and $\delta R$.

I reserve the formation of these developments and of the final equations for determining the coefficients of the different inequalities to another opportunity. These equations are voluminous when all sensible quantities are taken into account; but they are formed with so much facility by means of Table II., that error is not likely to arise in this part of the process. Error is more, I think, to be apprehended in the terms of $R$ multiplied by the cubes and fourth powers of the eccentricities; the rest have been verified by an independent method.

See p. 39.

## Addition to Table I.

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 1 | 148<br>147 | 153<br>152 | 154<br>151 | 1 |
| 2 | 150<br>149 | 161<br>162 | 162<br>−146 | 2 |
| 3 | 152<br>151 | 147<br>164 | 148<br>163 | 3 |
| 4 | 154<br>153 | 165<br>148 | 166<br>147 | 4 |
| 5 | 156<br>155 | 167<br>−173 | 168<br>−174 | 5 |
| 6 | 158<br>157 | 169<br>170 | 178<br>169 | 6 |
| 7 | 160<br>159 | 171<br>176 | 172<br>175 | 7 |
| 146 | 62<br>0 | 2<br>− 65 | 66<br>− 2 | 146 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 147 | 1<br>63 | 69<br>3 | 4<br>67 | 147 |
| 148 | 64<br>1 | 4<br>68 | 70<br>3 | 148 |
| 149 | 2<br>65 | 77<br>0 | 8<br>− 62 | 149 |
| 150 | 66<br>2 | 8<br>62 | 78<br>0 | 150 |
| 151 | 3<br>67 | 63<br>9 | 1<br>79 | 151 |
| 152 | 68<br>3 | 1<br>80 | 64<br>9 | 152 |
| 153 | 4<br>69 | 81<br>1 | 10<br>63 | 153 |
| 154 | 70<br>4 | 10<br>64 | 82<br>1 | 154 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 155 | 5<br>71 | 83<br>− 14 | 11<br>− 90 | 155 |
| 156 | 72<br>5 | 11<br>− 89 | 84<br>− 14 | 156 |
| 157 | 6<br>73 | 93<br>12 | 16<br>85 | 157 |
| 158 | 74<br>6 | 16<br>86 | 94<br>12 | 158 |
| 159 | 7<br>75 | 87<br>15 | 13<br>91 | 159 |
| 160 | 76<br>7 | 13<br>92 | 88<br>15 | 160 |

| | 161 | 162 | |
|---|---|---|---|
| | 165<br>164 | 166<br>163 | 1 |
| 146 | 8<br>− 77 | 78<br>− 8 | 146 |

| | 161 | 162 | |
|---|---|---|---|
| 147 | 81<br>9 | 10<br>79 | 147 |
| 148 | 10<br>80 | 82<br>9 | 148 |

## Addition to Table II.

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 1 | 147<br>148 | 152<br>153 | 151<br>154 | 1 |
| 2 | 149<br>150 | 146<br>..... | .....<br>−146 | 2 |
| 3 | 151<br>152 | .....<br>147 | .....<br>148 | 3 |
| 4 | 153<br>154 | 148<br>..... | 147<br>..... | 4 |
| 5 | 155<br>156 | .....<br>..... | .....<br>..... | 5 |
| 6 | 157<br>158 | .....<br>..... | .....<br>..... | 6 |
| 7 | 159<br>160 | .....<br>..... | .....<br>..... | 7 |
| 8 | 161<br>162 | 150<br>..... | 149<br>..... | 8 |
| 9 | 163<br>164 | .....<br>151 | .....<br>152 | 9 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 10 | 165<br>166 | 154<br>..... | 153<br>..... | 10 |
| 11 | 167<br>168 | 156<br>..... | 155<br>..... | 11 |
| 12 | 169<br>170 | .....<br>157 | .....<br>158 | 12 |
| 13 | 171<br>172 | 160<br>..... | 159<br>..... | 13 |
| 14 | 173<br>174 | .....<br>−155 | .....<br>−156 | 14 |
| 15 | 175<br>176 | .....<br>159 | .....<br>160 | 15 |
| 16 | 177<br>178 | 158<br>..... | 157<br>..... | 16 |
| 62 | 146<br>..... | .....<br>150 | .....<br>−149 | 62 |
| 63 | .....<br>147 | 151<br>..... | .....<br>153 | 63 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 64 | 148<br>..... | .....<br>154 | 152<br>..... | 64 |
| 65 | .....<br>149 | .....<br>−146 | .....<br>..... | 65 |
| 66 | 150<br>..... | .....<br>..... | 146<br>..... | 66 |
| 67 | .....<br>151 | .....<br>..... | .....<br>147 | 67 |
| 68 | 152<br>..... | .....<br>148 | .....<br>..... | 68 |
| 69 | .....<br>153 | 147<br>..... | .....<br>..... | 69 |
| 70 | 154<br>..... | .....<br>..... | 148<br>..... | 70 |
| 71 | .....<br>155 | .....<br>..... | .....<br>..... | 71 |
| 72 | 156<br>..... | .....<br>..... | .....<br>..... | 72 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 73 | ..... / 157 | ..... / ..... | ..... / ..... | 73 |
| 74 | 158 / ..... | ..... / ..... | ..... / ..... | 74 |
| 75 | ..... / 159 | ..... / ..... | ..... / ..... | 75 |
| 76 | 160 / ..... | ..... / ..... | ..... / ..... | 76 |
| 77 | ..... / 161 | 149 / ..... | ..... / ..... | 77 |
| 78 | 162 / ..... | ..... / ..... | 150 / ..... | 78 |
| 79 | ..... / 163 | ..... / ..... | ..... / 151 | 79 |
| 80 | 164 / ..... | ..... / 152 | ..... / ..... | 80 |
| 81 | ..... / 165 | 153 / ..... | ..... / ..... | 81 |
| 82 | 166 / ..... | ..... / ..... | 154 / ..... | 82 |
| 83 | ..... / 167 | 155 / ..... | ..... / ..... | 83 |
| 84 | 168 / ..... | ..... / ..... | 156 / ..... | 84 |
| 85 | ..... / 169 | ..... / ..... | ..... / 157 | 85 |
| 86 | 170 / ..... | ..... / 158 | ..... / ..... | 86 |
| 87 | ..... / 171 | 159 / ..... | ..... / ..... | 87 |
| 88 | 172 / ..... | ..... / ..... | 160 / ..... | 88 |
| 89 | ..... / 173 | ..... / −156 | ..... / ..... | 89 |
| 90 | 174 / ..... | ..... / ..... | ..... / −155 | 90 |
| 91 | ..... / 175 | ..... / ..... | ..... / 159 | 91 |
| 92 | 176 / ..... | ..... / 160 | ..... / ..... | 92 |
| 93 | ..... / 177 | 157 / ..... | ..... / ..... | 93 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 94 | 178 / ..... | ..... / ..... | 158 / ..... | 94 |
| 146 | ..... / 62 | ..... / ..... | ..... / − 9 | 146 |
| 147 | 63 / 1 | 3 / ..... | ..... / 4 | 147 |
| 148 | 1 / 64 | ..... / 4 | 3 / ..... | 148 |
| 149 | ..... / 2 | ..... / ..... | ..... / ..... | 149 |
| 150 | 2 / ..... | ..... / ..... | ..... / ..... | 150 |
| 151 | ..... / 3 | ..... / ..... | ..... / 1 | 151 |
| 152 | 3 / ..... | ..... / 1 | ..... / ..... | 152 |
| 153 | ..... / 4 | 1 / ..... | ..... / ..... | 153 |
| 154 | 4 / ..... | ..... / ..... | 1 / ..... | 154 |
| 155 | ..... / 5 | ..... / ..... | ..... / ..... | 155 |
| 156 | 5 / ..... | ..... / ..... | ..... / ..... | 156 |
| 157 | ..... / 6 | ..... / ..... | ..... / ..... | 157 |
| 158 | 6 / ..... | ..... / ..... | ..... / ..... | 158 |
| 159 | ..... / 7 | ..... / ..... | ..... / ..... | 159 |
| 160 | 7 / ..... | ..... / ..... | ..... / ..... | 160 |
| 161 | ..... / 8 | 2 / ..... | ..... / ..... | 161 |
| 162 | 8 / ..... | ..... / 2 | 2 / ..... | 162 |
| 163 | ..... / 9 | ..... / ..... | ..... / 3 | 163 |
| 164 | 9 / ..... | ..... / 3 | ..... / ..... | 164 |
| 165 | ..... / 10 | 4 / ..... | ..... / ..... | 165 |

| | 146 | 149 | 150 | |
|---|---|---|---|---|
| 166 | 10 / ..... | ..... / ..... | 4 / ..... | 166 |
| 167 | ..... / 11 | 5 / ..... | ..... / ..... | 167 |
| 168 | 11 / ..... | ..... / ..... | 5 / ..... | 168 |
| 169 | ..... / 12 | 6 / ..... | ..... / 6 | 169 |
| 170 | 12 / ..... | ..... / 6 | ..... / ..... | 170 |
| 171 | ..... / 13 | 7 / ..... | ..... / ..... | 171 |
| 172 | 13 / ..... | ..... / ..... | 7 / ..... | 172 |
| 173 | ..... / 14 | − 5 / ..... | ..... / ..... | 173 |
| 174 | 14 / ..... | ..... / ..... | ..... / − 5 | 174 |
| 175 | ..... / 15 | ..... / ..... | ..... / 7 | 175 |
| 176 | 15 / ..... | ..... / 7 | ..... / ..... | 176 |
| 177 | ..... / 16 | ..... / ..... | ..... / ..... | 177 |
| 178 | 16 / ..... | ..... / ..... | 6 / ..... | 178 |
| 179 | ..... / 17 | ..... / ..... | ..... / ..... | 179 |
| 180 | 17 / ..... | ..... / ..... | ..... / ..... | 180 |
| 181 | ..... / 18 | ..... / ..... | ..... / ..... | 181 |
| 182 | 18 / ..... | ..... / ..... | ..... / ..... | 182 |
| 183 | ..... / 19 | ..... / ..... | ..... / ..... | 183 |
| 184 | 19 / ..... | ..... / ..... | ..... / ..... | 184 |

| | 161 | 162 | |
|---|---|---|---|
| 8 | 146 / ..... | ..... / −146 | 8 |
| 9 | ..... / 147 | ..... / 148 | 9 |
| 10 | 148 / ..... | 147 / ..... | 10 |
| 77 | ..... / −146 | ..... / ..... | 77 |

| | 161 | 162 | |
|---|---|---|---|
| 78 | ..... / ..... | 146 / ..... | 78 |
| 79 | ..... / ..... | ..... / 147 | 79 |
| 80 | ..... / 148 | ..... / ..... | 80 |

| | 161 | 162 | |
|---|---|---|---|
| 81 | 147 / ..... | ..... / ..... | 81 |
| 82 | ..... / ..... | 148 / ..... | 82 |
| 163 | ..... / ..... | ..... / 1 | 163 |

| | 161 | 162 | |
|---|---|---|---|
| 164 | ..... / 1 | ..... / ..... | 164 |
| 165 | 1 / ..... | ..... / ..... | 165 |
| 166 | ..... / ..... | 1 / ..... | 166 |

*On the Precession of the Equinoxes, supposing the Earth to revolve in a resisting medium.*

In my last paper on Physical Astronomy, I gave expressions for the variations of the six constants which enter into the solution of this problem, upon the hypothesis that the body revolves in a medium devoid of resistance.

If we suppose a plane to revolve in a resisting medium, about an axis perpendicular to itself, the resistance of the medium can produce no effect, and the phenomena will only be modified in a slight degree by the friction of the plane surface against the medium. If, however, the inclination of the plane on the axis of rotation differs from 90°, the effect of the resistance of the medium becomes sensible, tending to retard the motion of the plane; the effect being greatest when the axis of rotation is parallel to the plane.

This principle is used in some machines, as in self-playing organs, to regulate the motion by means of a vane, of which the inclination to its axis of rotation can be varied at pleasure.

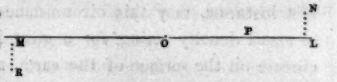
In the case of a sphere, whatever be the direction of the axis of rotation, this effect of the resistance is insensible; and also in the case of a solid of revolution when the axis of rotation coincides with the axis of the figure, but not otherwise. If the difference of the latitude of the axis of rotation from 90° (supposing the equator from which the latitudes are reckoned to coincide with the equator of the figure) be at any time small, the mathematical investigation appears to show, that the effect of the resistance of the medium is to diminish continually this difference. In the case of the earth, this quantity is now insensible; but as the probability is small that this was the case in the first instance, may this circumstance arise from the resistance of a medium of small density acting for a great length of time? and can the change of climate on the surface of the earth, a change of which the probability is indicated by many geological phenomena, be explained in the same manner? It may be remarked, however, that the effect of a resisting medium in diminishing the eccentricities of the orbits of the planets is of the same order, and that these, although for the most part small, are far from having reached zero. The tendency of a resisting medium is also to diminish the major axes of the orbits of the planets; these effects, if they exist, will probably be most sensible

in the case of comets, not only on account of their great eccentricity, but also on account of their small density, in the same manner as a flock of any light substance is wafted by the gentlest wind and prevented from reaching the ground. The eccentricity of the orbit of the comet of Halley in 1759 is known with great accuracy, and as its perturbations have been calculated with great care by MM. Damoiseau and de Pontécoulant, the eccentricity which it should have in 1835, when it will again visit this part of space, unless it be affected by a resisting medium, is also known with great precision. It is scarcely probable, however, that any change will be perceptible in one revolution, even if the cause exists; but the succeeding revolutions of this body will no doubt throw light upon this question. The ratio of the change of the semi-major axis to the change of the eccentricity, due to the action of the resisting medium, is known, being a function of the eccentricity, and independent of the constant, which depends upon the density of the medium; this ratio therefore may also tend to elucidate the question, if it can be determined by observation with sufficient accuracy.

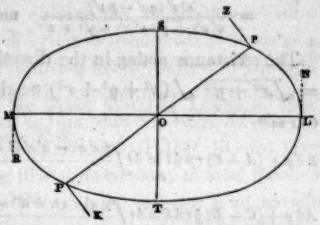
Let $x'$, $y'$, $z'$ be the co-ordinates of any point P corresponding to the elementary portion of the surface $ds$, and referred to axes passing through the centre of gravity and revolving with the body in motion.

Let P be the point of which the co-ordinates are $x'$, $y'$, $z'$, A P the direction of the normal at the point P, B P perpendicular to the axis of instantaneous rotation, and cutting it in B, and C P the direction of motion of the point P. I suppose the resistance of the medium to create a force proportional to $v^2 \cos \mathrm{A\,P\,C}\, ds$, acting in the direction of the normal A P upon the point P, $v$ being the velocity of the point P.

Suppose the straight line M O P L to revolve about an axis passing through O,




and perpendicular to it, and in the direction L N, the action of the resisting medium will be in the direction N L, on one side only of the line O L, upon all the points P between O and L, and upon all the points between M P it will be in the contrary direction R M, and on the other side of the line.




Now, let L S M T be the section of a cylinder revolving about an axis, passing through O perpendicular to the plane L S M T, and let the cylinder revolve in the direction L N. The action of the resisting medium will be in the direction Z P, perpendicular to O P upon all the points P between L S; and in the contrary direction K P upon all the points, P between T M. These remarks show that in what follows, the integrations must not be made throughout the whole surface of the body revolving: this consideration however does not affect the nature of the results.

The equation to a plane perpendicular to the axis of rotation, and passing through the centre of gravity of the body, is $p\,x + q\,y + r\,z = 0$.

Let the body revolving be a spheroid of which the equation is

$$x^2 + y^2 + z^2\,(1 + e^2) = a^2\,(1 + e^2)$$

The equation to the tangent plane to the spheroid at the point $x$, $y$, $z$ is

$$x\,x' + y\,y' + z\,z'\,(1 + e^2) = a^2\,(1 + e^2)$$

The equations to the planes from whose intersection the line P B results, are

$$* \; z\,(q\,x' - r\,y') + y\,(r\,x' - p\,z') + z\,(p\,y' - q\,x') = 0$$
$$p\,x + q\,y + r\,z = D$$

$D$ being a constant. The equations to the line P C are

$$x\{r\,(q\,x' - r\,y') - p\,(p\,y' - q\,x')\} + y\{r\,(r\,x' - p\,z') - q\,(p\,y' - q\,x')\} = 0$$
$$x\{q\,(q\,x' - r\,y') - p\,(r\,x' - p\,z')\} + z\{q\,(p\,y' - q\,x') - r\,(r\,x' - p\,z')\} = 0$$

and neglecting $p^2$, $q^2$, $p\,q$,

$$x\,(q\,x' - r\,y') = y\,(p\,x' - r\,z')$$
$$x\,(q\,y' + p\,x') = z\,(p\,x' - r\,z')$$

The equations to the direction of motion of the point P are

$$x\,(p\,x' - r\,z') = y\,(r\,y' - q\,r')$$
$$x\,(q\,x' - p\,y') = z\,(r\,y' - q\,z')$$

Cos. angle, which the direction of motion of P makes with the normal to the surface or cos A P C

$$= \frac{x'\,(r\,y' - q\,x') + y'\,(p\,z' - r\,x') + z'\,(1 + e^2)\,(q\,x' - p\,y')}{\sqrt{\{(r\,y' - q\,x')^2 + (p\,z' - r\,x')^2 + (q\,y - p\,x')^2\}}\;\sqrt{\{x'^2 + y'^2 + z'^2\,(1 + 2\,e^2)\}}}$$

* The notation is the same as p. 20, except that the accents at foot of $x_1$, $y_1$, $z_1$, are omitted.

$$= \frac{e^2 x' (q x' - p y')}{r \sqrt{x'^2 + y'^2} \sqrt{x'^2 + y'^2 + z'^2}} \quad \text{nearly.}$$

The resistance acting in the direction of the normal, and since the velocity $= \sqrt{x'^2 + y'^2}\ \sqrt{(p^2 + q^2 + r^2)}$ nearly;

$$C\,dr = 0$$

$$B\,dq + (A - C)\,r\,p\,dt = dt \int \frac{\{x' x' - x' x' (1 + e^2)\}\, e^2 z' (q x' - p y') \sqrt{x'^2 + y'^2}\, ds\, (p^2 + q^2 + r^2)}{r \{x'^2 + y'^2 + z'^2\}}$$

$$A\,dp + (C - B)\,q\,r\,dt = dt \int \frac{\{y' z' (1 + e^2) - z' y'\}\, e^2 z' (q x' - p y') \sqrt{x'^2 + y'^2}\, ds\, (p^2 + q^2 + r^2)}{r \{x'^2 + y'^2 + z'^2\}}$$

$$\sin \frac{C - A}{A} (n t + \gamma)\, dc + c \frac{(C - A)}{A} \cos \frac{C - A}{A} (n t + \gamma)\, d\gamma$$

$$= - \frac{n\, dt\, e^2}{A} \int \frac{x' z' (q x' - p y') \sqrt{x'^2 + y'^2}\, ds}{\{x'^2 + y'^2 + z'^2\}}$$

$$\cos \frac{C - A}{A} (n t + \gamma)\, dc - c \frac{(C - A)}{A} \sin \frac{C - A}{A} (n t + \gamma)\, d\gamma$$

$$= \frac{n\, dt\, e^2}{A} \int \frac{y' z' (q x' - p y') \sqrt{x'^2 + y'^2}\, ds}{\{x'^2 + y'^2 + z'^2\}}$$

since $\int x'^2 z'^2\, ds = \int y'^2 z'^2\, ds$

$$dc = - \frac{n\, dt\, e^2 c}{A} \int \frac{x'^2 z'^2 \sqrt{x'^2 + y'^2}\, ds}{\{x'^2 + y'^2 + z'^2\}} + \frac{n\, dt\, e^2}{2A} \sin 2 \frac{(C - A)}{A} (n t + \gamma) \int \frac{x' y' z'^2 \sqrt{x'^2 + y'^2}\, ds}{\{x'^2 + y'^2 + z'^2\}}$$

neglecting the term which is periodic,

$$dc = - n c \frac{e^2\, dt}{A} \int \frac{x'^2 z'^2 \sqrt{x'^2 + y'^2}\, ds}{\{x'^2 + y'^2 + z'^2\}}$$

Let $\int \frac{x'^2 z'^2 \sqrt{x'^2 + y'^2}\, ds}{\{x'^2 + y'^2 + z'^2\}} = D$

$D$ being a positive quantity.

$$dc = - \frac{n D c e^2\, dt}{A} \qquad e^{\frac{1}{c}} = \frac{n D e^2 t}{A},$$ $e$ being the base of Naperian logarithms.

When $t$ is infinite $c = 0$; hence the latitude of the axis of instantaneous rotation increases until it reaches 90°, which is its limit.

Having determined the variations of $c$, $\gamma$ and $n$ by means of the above equations, the variations of the other constants $\omega$, $\psi_0$ and $\varphi_0$ may be determined from the equations

$$p\,dt = \sin \varphi \sin \theta\, d\psi - \cos \varphi\, d\theta$$
$$q\,dt = \cos \varphi \sin \theta\, d\psi + \sin \varphi\, d\theta$$
$$r\,dt = d\varphi - \cos \theta\, d\psi$$

# XVI. *Researches in Physical Astronomy.* *By* John William Lubbock, *Esq. V. P. and Treas. R.S.*

Read June 9, 1831.

I PROPOSE in this paper to extend the equations I have already given for determining the planetary inequalities, as far as the terms depending on the squares and products of the eccentricities, to the terms depending on the cubes of the eccentricities and quantities of that order, which is done very easily by a Table similar to Table II. in my Lunar Theory; and particularly to the determination of the great inequality of Jupiter, or at least such part of it as depends on the first power of the disturbing force. That part which depends on the square of the disturbing force may I think be most easily calculated by the methods given in my Lunar Theory; but not without great care and attention can accurate numerical results be expected. I have however given the analytical form of the coefficients of the arguments in the development of $R$, upon which that inequality principally depends.

It is I think particularly convenient to designate the arguments of the planetary disturbances by indices. The system of indices adopted in this paper is given as appearing better adapted for the purpose than that used in my former paper on the Planetary Theory; but it is not advisable to make use of the same indices in this as in the Lunar Theory.

I have also given analytical expressions for the development of $R$ to the terms multiplied by the squares and products of the eccentricities inclusive, and for the terms in $r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$ multiplied by the first power of the eccentricities, which are I believe the simplest that can be proposed.

The following are the arguments which occur in the Planetary Theory.

Column 1 contains the index.

—— 2 contains the index of the argument, which is symmetrical.

—— 3 contains the index used Phil. Trans. Part II. 1830, p. 349.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | .. | .. | $0$ | 104 | .. | 39 | $4t+x-z=5nt-5n_1t-\varpi+\varpi_1$ |
| 1 | .. | .. | $t=nt-n_1t$ | 110 | 50 | 57 | $2z=2n_1t-2\varpi_1$ |
| 2 | .. | .. | $2t=2nt-2n_1t$ | 111 | 61 | 63 | $t-2z=nt-3n_1t+2\varpi_1$ |
| 3 | .. | .. | $3t=3nt-3n_1t$ | 112 | 62 | 64 | $2t-2z=2nt-4n_1t+2\varpi_1$ |
| 4 | .. | .. | $4t=4nt-4n_1t$ | 113 | 63 | 65 | $3t-2z=3nt-5n_1t+2\varpi_1$ |
| 10 | 30 | 7 | $x=nt-\varpi$ | 114 | 64 | 66 | $4t-2z=4nt-6n_1t+2\varpi_1$ |
| 11 | 41 | 6 | $t-x=-n_1t+\varpi$ | 121 | 51 | 58 | $t+2x=nt+n_1t-2\varpi_1$ |
| 12 | 42 | 12 | $2t-x=nt-2n_1t+\varpi$ | 122 | 52 | 59 | $2t+2z=2nt-2\varpi_1$ |
| 13 | 43 | 13 | $3t-x=2nt-3n_1t+\varpi$ | 123 | 52 | 60 | $3t+2z=3nt-n_1t-2\varpi_1$ |
| 14 | 44 | 14 | $4t-x=3nt-4n_1t+\varpi$ | 124 | 54 | 61 | $4t+2z=4nt-2n_1t-2\varpi_1$ |
| 21 | 31 | 8 | $t+x=2nt-n_1t-\varpi$ | 130 | .. | 69 | $2y=2n_1t-2\nu_1$ |
| 22 | 32 | 9 | $2t+x=3nt-2n_1t-\varpi$ | 131 | .. | 71 | $t-2y=nt-3n_1t+2\nu_1$ |
| 23 | 33 | 10 | $3t+x=4nt-3n_1t-\varpi$ | 132 | .. | 73 | $2t-2y=2nt-4n_1t+2\nu_1$ |
| 24 | 34 | 11 | $4t+x=5nt-4n_1t-\varpi$ | 133 | .. | .. | $3t-2y=3nt-5n_1t+2\nu_1$ |
| 30 | 10 | 15 | $z=n_1t-\varpi_1$ | 134 | .. | .. | $4t-2y=4nt-6n_1t+2\nu_1$ |
| 31 | 21 | 20 | $t-z=nt-2n_1t+\varpi_1$ | 141 | .. | 68 | $t+2y=nt+n_1t-2\nu_1$ |
| 32 | 22 | 21 | $2t-z=2nt-3n_1t+\varpi_1$ | 142 | .. | 70 | $2t+2y=2nt-2\nu_1$ |
| 33 | 23 | 22 | $3t-z=3nt-4n_1t+\varpi_1$ | 143 | .. | 72 | $3t+2y=3nt-n_1t-2\nu_1$ |
| 34 | 24 | 23 | $4t-z=4nt-5n_1t+\varpi_1$ | 144 | .. | .. | $4t+2y=4nt-2n_1t-2\nu_1$ |
| 41 | 11 | 16 | $t+z=nt-\varpi_1$ | 150 | 250 | .. | $3x=3nt-3\varpi$ |
| 42 | 12 | 17 | $2t+z=2nt-n_1t-\varpi_1$ | 151 | 261 | .. | $t-3x=-2nt-n_1t+3\varpi$ |
| 43 | 13 | 18 | $3t+z=3nt-2n_1t-\varpi_1$ | 152 | 262 | .. | $2t-3x=-nt-2n_1t+3\varpi$ |
| 44 | 14 | 19 | $4t+z=4nt-3n_1t-\varpi_1$ | 153 | 263 | .. | $3t-3x=-3n_1t+3\varpi$ |
| 50 | 110 | 26 | $2x=2nt-2\varpi$ | 154 | 264 | .. | $4t-3x=nt-4n_1t+3\varpi$ |
| 51 | 121 | 25 | $t-2x=-nt-n_1t+2\varpi$ | 161 | 251 | .. | $t+3x=4nt-n_1t-3\varpi$ |
| 52 | 122 | 24 | $2t-2x=-2n_1t+2\varpi$ | 162 | 252 | .. | $2t+3x=5nt-2n_1t-3\varpi$ |
| 53 | 123 | 32 | $3t-2x=nt-3n_1t-2\varpi$ | 163 | 253 | .. | $3t+3x=6nt-3n_1t-3\varpi$ |
| 54 | 124 | 33 | $4t-2x=2nt-4n_1t+2\varpi$ | 164 | 254 | .. | $4t+3x=7nt-4n_1t-3\varpi$ |
| 61 | 111 | 27 | $t+2x=3nt-n_1t-2\varpi$ | 170 | 210 | .. | $2x+z=2nt+n_1t-2\varpi-\varpi_1$ |
| 62 | 112 | 28 | $2t+2x=4nt-2n_1t-2\varpi$ | 171 | 221 | .. | $t-2x-z=-nt-2n_1t+2\varpi+\varpi_1$ |
| 63 | 113 | 29 | $3t+2x=5nt-3n_1t-2\varpi$ | 172 | 222 | .. | $2t-2x-z=-3n_1t+2\varpi+\varpi_1$ |
| 64 | 114 | 30 | $4t+2x=6nt-4n_1t-2\varpi$ | 173 | 223 | .. | $3t-2x-z=nt-4n_1t+2\varpi+\varpi_1$ |
| 70 | .. | 47 | $x+z=nt+n_1t-\varpi-\varpi_1$ | 174 | 224 | .. | $4t-2x-z=2nt-5n_1t+2\varpi+\varpi_1$ |
| 71 | 81 | 46 | $t-x-z=-2n_1t+\varpi+\varpi_1$ | 181 | 211 | .. | $t+2x+z=3nt-2\varpi-\varpi_1$ |
| 72 | 82 | 53 | $2t-x-z=nt-3n_1t+\varpi+\varpi_1$ | 182 | 212 | .. | $2t+2x+z=4nt-n_1t-2\varpi-\varpi_1$ |
| 73 | 83 | 54 | $3t+x-z=2nt-4n_1t+\varpi+\varpi_1$ | 183 | 213 | .. | $3t+2x+z=5nt-2n_1t-2\varpi-\varpi_1$ |
| 74 | 84 | 55 | $4t-x-z=3nt-5n_1t+\varpi+\varpi_1$ | 184 | 214 | .. | $4t+2x+z=6nt-3n_1t-2\varpi-\varpi_1$ |
| 81 | 71 | 48 | $t+x+z=2nt-\varpi-\varpi_1$ | 190 | 230 | .. | $2x-z=2nt-n_1t-2\varpi+\varpi_1$ |
| 82 | 72 | 49 | $2t+x+z=3nt-n_1t-\varpi-\varpi_1$ | 191 | 231 | .. | $t-2x+z=-nt+2\varpi-\varpi_1$ |
| 83 | 73 | 50 | $3t+x+z=4nt-2n_1t-\varpi-\varpi_1$ | 192 | 232 | .. | $2t-2x+z=-n_1t+2\varpi-\varpi_1$ |
| 84 | 74 | 51 | $4t+x+z=5nt-3n_1t-\varpi-\varpi_1$ | 193 | 233 | .. | $3t-2x+z=nt-2n_1t+2\varpi-\varpi_1$ |
| 90 | .. | 35 | $x-z=nt-n_1t-\varpi+\varpi_1$ | 194 | 234 | .. | $4t-2x+z=2nt-3n_1t+2\varpi-\varpi_1$ |
| 91 | .. | 41 | $t-x+z=\varpi-\varpi_1$ | 201 | 241 | .. | $t+2x-z=3nt-2n_1t-2\varpi+\varpi_1$ |
| 92 | .. | 42 | $2t-x+z=nt-n_1t+\varpi-\varpi_1$ | 202 | 242 | .. | $2t+2x-z=4nt-3n_1t-2\varpi+\varpi_1$ |
| 93 | .. | 43 | $3t-x+z=2nt-2n_1t+\varpi+\varpi_1$ | 203 | 243 | .. | $3t+2x-z=5nt-4n_1t-2\varpi+\varpi_1$ |
| 94 | .. | 44 | $4t-x+z=3nt-3n_1t+\varpi+\varpi_1$ | 204 | 244 | .. | $4t+2x-z=6nt-5n_1t-2\varpi+\varpi_1$ |
| 101 | .. | 36 | $t+x-z=2nt-2n_1t-\varpi+\varpi_1$ | 210 | 170 | .. | $x+2z=nt+2n_1t-\varpi-2\varpi_1$ |
| 102 | .. | 37 | $2t+x-z=3nt-3n_1t-\varpi+\varpi_1$ | 211 | 222 | .. | $t-x-2z=-3n_1t+\varpi+2\varpi_1$ |
| 103 | .. | 38 | $3t+x-z=4nt-4n_1t-\varpi+\varpi_1$ | 212 | 223 | .. | $2t-x-2z=nt-4n_1t+\varpi+2\varpi_1$ |

| | | | | | |
|---|---|---|---|---|---|
| 213 | 224 | $3t-x-2z=2nt-5n_it+\varpi+2\varpi_i$ | 282 | .. | $2t+x+2y=3nt-\varpi-2\nu_i$ |
| 214 | 225 | $4t-x-2z=3nt-6n_it+\varpi+2\varpi_i$ | 283 | .. | $3t+x+2y=4nt-n_it-\varpi-2\nu_i$ |
| 221 | 171 | $t+x+2z=2nt+n_it-\varpi-2\varpi_i$ | 284 | .. | $4t+x+2y=5nt-2n_it-\varpi-2\nu_i$ |
| 222 | 172 | $2t+x+2z=3nt-\varpi-2\varpi_i$ | 290 | .. | $x-2y=nt-2n_it-\varpi+2\nu_i$ |
| 223 | 173 | $3t+x+2z=4nt-n_it-\varpi-2\varpi_i$ | 291 | .. | $t-x+2y=n_it+\varpi-2\nu_i$ |
| 224 | 174 | $4t+x+2z=5nt-n_it-\varpi-2\varpi_i$ | 292 | .. | $2t-x+2y=nt+\varpi-2\nu_i$ |
| 230 | 190 | $x-2z=nt-2n_it-\varpi+2\varpi_i$ | 293 | .. | $3t-x+2y=2nt-n_it+\varpi-2\nu_i$ |
| 231 | 191 | $t-x+2z=n_it+\varpi-2\varpi_i$ | 294 | .. | $4t-x+2y=3nt-2n_it+\varpi-2\nu_i$ |
| 232 | 192 | $2t-x+2z=nt+\varpi-2\varpi_i$ | 301 | .. | $t+x-2y=2nt-3n_it-\varpi+2\nu_i$ |
| 233 | 193 | $3t-x+2z=2nt-n_it+\varpi-2\varpi_i$ | 302 | .. | $2t+x-2y=3nt-4n_it-\varpi+2\nu_i$ |
| 234 | 194 | $4t-x+2z=3nt-2n_it+\varpi-2\varpi_i$ | 303 | .. | $3t+x-2y=4nt-5n_it-\varpi+2\nu_i$ |
| 241 | 201 | $t+x-2z=2nt-3n_it-\varpi+2\varpi_i$ | 304 | .. | $4t+x-2y=5nt-6n_it-\varpi+2\nu_i$ |
| 242 | 202 | $2t+x-2z=3nt-4n_it-\varpi+2\varpi_i$ | 310 | .. | $z+2y=3n_it-\varpi_i-2\nu_i$ |
| 243 | 203 | $3t+x-2z=4nt-5n_it-\varpi+2\varpi_i$ | 311 | .. | $t-z-2y=nt-4n_it+\varpi_i+2\nu_i$ |
| 244 | 204 | $4t+x-2z=5nt-6n_it-\varpi+2\varpi_i$ | 312 | .. | $2t-z-2y=2nt-5n_it+\varpi_i+2\nu_i$ |
| 250 | 150 | $3z=3n_it-3\varpi_i$ | 313 | .. | $3t-z-2y=3nt-6n_it+\varpi_i+2\nu_i$ |
| 251 | 161 | $t-3z=nt-4n_it+3\varpi_i$ | 314 | .. | $4t-z-2y=4n_it-7n_it+\varpi_i+2\nu_i$ |
| 252 | 162 | $2t-3z=2nt-5n_it+3\varpi_i$ | 321 | .. | $t+z+2y=nt+2n_it-\varpi_i-2\nu_i$ |
| 253 | 163 | $3t-3z=3nt-6n_it+3\varpi_i$ | 322 | .. | $2t+z+2y=2nt+n_it-\varpi_i-2\nu_i$ |
| 254 | 164 | $4t-3z=4nt-7n_it+3\varpi_i$ | 323 | .. | $3t+z+2y=3nt-\varpi_i-2\nu_i$ |
| 261 | 151 | $t+3z=nt+2n_it-3\varpi_i$ | 324 | .. | $4t+z+2y=4nt-n_it-\varpi_i-2\nu_i$ |
| 262 | 152 | $2t+3z=2n+n_it-3\varpi_i$ | 330 | .. | $z-2y=-n_it-\varpi_i+2\nu_i$ |
| 263 | 153 | $3t+3z=3nt-3\varpi_i$ | 331 | .. | $t-z+2y=nt+\varpi_i-2\nu_i$ |
| 264 | 154 | $4t+3z=4nt-n_it-3\varpi_i$ | 332 | .. | $2t-z+2y=2nt-n_it+\varpi_i-2\nu_i$ |
| 270 | .. | $x+2y=nt+2n_it-\varpi-2\nu_i$ | 333 | .. | $3t-z+2y=3nt-2n_it+\varpi_i+2\nu_i$ |
| 271 | .. | $t-x-2y=-3n_it+\varpi+2\nu_i$ | 334 | .. | $4t-z+2y=4nt-3n_it+\varpi_i-2\nu_i$ |
| 272 | .. | $2t-x-2y=nt-4n_it+\varpi+2\nu_i$ | 341 | .. | $t+z-2y=nt-2n_it-\varpi_i+2\nu_i$ |
| 273 | .. | $3t-x-2y=2nt-5n_it+\varpi+2\nu_i$ | 342 | .. | $2t+z-2y=2nt-3n_it-\varpi_i+2\nu_i$ |
| 274 | .. | $4t-x-2y=3nt-6n_it+\varpi+2\nu_i$ | 343 | .. | $3t+z-2y=3nt-4n_it-\varpi_i+2\nu_i$ |
| 281 | .. | $t+x+2y=2nt+n_it-\varpi-2\nu_i$ | 344 | .. | $4t+z-2y=4nt-5n_it-\varpi_i+2\nu_i$ |

## Table I.

Showing the arguments which result from the combination of the arguments 10, 50 and 150 with the arguments in the first or left-hand column, by addition and subtraction.

| | 10 | 50 | 150 | |
|---|---|---|---|---|
| 1 | 21<br>11 | 61<br>51 | 161<br>151 | 1 |
| 2 | 22<br>12 | 62<br>52 | 162<br>152 | 2 |
| 3 | 23<br>13 | 63<br>53 | 163<br>153 | 3 |
| 4 | 24<br>14 | 64<br>54 | 164<br>154 | 4 |
| 10 | 50<br>0 | 150<br>− 10 | ……<br>…… | 10 |
| 11 | 1<br>51 | 21<br>151 | ……<br>…… | 11 |
| 12 | 2<br>52 | 22<br>152 | ……<br>…… | 12 |
| 13 | 3<br>53 | 23<br>153 | ……<br>…… | 13 |
| 14 | 4<br>54 | 24<br>154 | ……<br>…… | 14 |
| 21 | 61<br>1 | 161<br>11 | ……<br>…… | 21 |
| 22 | 62<br>2 | 162<br>12 | ……<br>…… | 22 |
| 23 | 63<br>3 | 163<br>13 | ……<br>…… | 23 |
| 24 | 64<br>4 | 164<br>14 | ……<br>…… | 24 |
| 30 | − 70<br>− 90 | 170<br>−190 | ……<br>…… | 30 |
| 31 | 101<br>71 | 201<br>171 | ……<br>…… | 31 |
| 32 | 102<br>72 | 202<br>172 | ……<br>…… | 32 |
| 33 | 103<br>73 | 203<br>173 | ……<br>…… | 33 |
| 34 | 104<br>74 | 204<br>174 | ……<br>…… | 34 |
| 41 | 81<br>91 | 181<br>191 | ……<br>…… | 41 |
| 42 | 82<br>92 | 182<br>192 | ……<br>…… | 42 |
| 43 | 83<br>93 | 183<br>193 | ……<br>…… | 43 |
| 44 | 84<br>94 | 184<br>194 | ……<br>…… | 44 |
| 50 | 150<br>10 | ……<br>…… | ……<br>…… | 50 |

| | 10 | 50 | 150 | |
|---|---|---|---|---|
| 51 | 11<br>151 | ……<br>…… | ……<br>…… | 51 |
| 52 | 12<br>152 | ……<br>…… | ……<br>…… | 52 |
| 53 | 13<br>153 | ……<br>…… | ……<br>…… | 53 |
| 54 | 14<br>154 | ……<br>…… | ……<br>…… | 54 |
| 61 | 161<br>21 | ……<br>…… | ……<br>…… | 61 |
| 62 | 162<br>22 | ……<br>…… | ……<br>…… | 62 |
| 63 | 163<br>23 | ……<br>…… | ……<br>…… | 63 |
| 64 | 164<br>24 | ……<br>…… | ……<br>…… | 64 |
| 70 | 170<br>30 | ……<br>…… | ……<br>…… | 70 |
| 71 | 31<br>171 | ……<br>…… | ……<br>…… | 71 |
| 72 | 32<br>172 | ……<br>…… | ……<br>…… | 72 |
| 73 | 33<br>173 | ……<br>…… | ……<br>…… | 73 |
| 74 | 34<br>174 | ……<br>…… | ……<br>…… | 74 |
| 81 | 181<br>41 | ……<br>…… | ……<br>…… | 81 |
| 82 | 182<br>42 | ……<br>…… | ……<br>…… | 82 |
| 83 | 183<br>43 | ……<br>…… | ……<br>…… | 83 |
| 84 | 184<br>44 | ……<br>…… | ……<br>…… | 84 |
| 90 | 190<br>− 30 | ……<br>…… | ……<br>…… | 90 |
| 91 | 41<br>191 | ……<br>…… | ……<br>…… | 91 |
| 92 | 42<br>192 | ……<br>…… | ……<br>…… | 92 |
| 93 | 43<br>193 | ……<br>…… | ……<br>…… | 93 |
| 94 | 44<br>194 | ……<br>…… | ……<br>…… | 94 |
| 101 | 201<br>31 | ……<br>…… | ……<br>…… | 101 |

| | 10 | 50 | 150 | |
|---|---|---|---|---|
| 102 | 202<br>32 | ……<br>…… | ……<br>…… | 102 |
| 103 | 203<br>33 | ……<br>…… | ……<br>…… | 103 |
| 104 | 204<br>34 | ……<br>…… | ……<br>…… | 104 |
| 110 | 210<br>−230 | ……<br>…… | ……<br>…… | 110 |
| 111 | 241<br>211 | ……<br>…… | ……<br>…… | 111 |
| 112 | 242<br>212 | ……<br>…… | ……<br>…… | 112 |
| 113 | 243<br>213 | ……<br>…… | ……<br>…… | 113 |
| 114 | 244<br>214 | ……<br>…… | ……<br>…… | 114 |
| 121 | 221<br>231 | ……<br>…… | ……<br>…… | 121 |
| 122 | 222<br>232 | ……<br>…… | ……<br>…… | 122 |
| 123 | 223<br>233 | ……<br>…… | ……<br>…… | 123 |
| 124 | 224<br>234 | ……<br>…… | ……<br>…… | 124 |
| 130 | 270<br>−290 | ……<br>…… | ……<br>…… | 130 |
| 131 | 301<br>271 | ……<br>…… | ……<br>…… | 131 |
| 132 | 302<br>272 | ……<br>…… | ……<br>…… | 132 |
| 133 | 303<br>273 | ……<br>…… | ……<br>…… | 133 |
| 134 | 304<br>274 | ……<br>…… | ……<br>…… | 134 |
| 141 | 281<br>291 | ……<br>…… | ……<br>…… | 141 |
| 142 | 282<br>292 | ……<br>…… | ……<br>…… | 142 |
| 143 | 283<br>293 | ……<br>…… | ……<br>…… | 143 |
| 144 | 284<br>294 | ……<br>…… | ……<br>…… | 144 |

## Table II.

Showing the arguments which, by their combination with the arguments 10, 50, and 150, by addition and subtraction, produce the arguments in the first or left-hand column.

| | 10 | 50 | 150 | | | 10 | 50 | 150 | | | 10 | 50 | 150 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 11<br>21 | .....<br>..... | .....<br>..... | 1 | 43 | 93<br>83 | .....<br>..... | .....<br>..... | 43 | 90 | .....<br>– 30 | .....<br>..... | .....<br>..... | 90 |
| 2 | 12<br>22 | .....<br>..... | .....<br>..... | 2 | 44 | 94<br>84 | .....<br>..... | .....<br>..... | 44 | 91 | .....<br>41 | .....<br>..... | .....<br>..... | 91 |
| 3 | 13<br>23 | .....<br>..... | .....<br>..... | 3 | 50 | 10<br>..... | 0<br>..... | .....<br>..... | 50 | 92 | .....<br>42 | .....<br>..... | .....<br>..... | 92 |
| 4 | 14<br>24 | .....<br>..... | .....<br>..... | 4 | 51 | .....<br>11 | .....<br>1 | .....<br>..... | 51 | 93 | .....<br>43 | .....<br>..... | .....<br>..... | 93 |
| 10 | 0<br>50 | .....<br>– 10 | .....<br>..... | 10 | 52 | .....<br>12 | .....<br>2 | .....<br>..... | 52 | 94 | .....<br>44 | .....<br>..... | .....<br>..... | 94 |
| 11 | 51<br>1 | .....<br>21 | .....<br>..... | 11 | 53 | .....<br>13 | .....<br>3 | .....<br>..... | 53 | 101 | 31<br>..... | .....<br>..... | .....<br>..... | 101 |
| 12 | 52<br>2 | .....<br>22 | .....<br>..... | 12 | 54 | .....<br>14 | .....<br>4 | .....<br>..... | 54 | 102 | 32<br>..... | .....<br>..... | .....<br>..... | 102 |
| 13 | 53<br>3 | .....<br>23 | .....<br>..... | 13 | 61 | 21<br>..... | 1<br>..... | .....<br>..... | 61 | 103 | 33<br>..... | .....<br>..... | .....<br>..... | 103 |
| 14 | 54<br>4 | .....<br>24 | .....<br>..... | 14 | 62 | 22<br>..... | 2<br>..... | .....<br>..... | 62 | 104 | 34<br>..... | .....<br>..... | .....<br>..... | 104 |
| 21 | 1<br>61 | 11<br>..... | .....<br>..... | 21 | 63 | 23<br>..... | 3<br>..... | .....<br>..... | 63 | 150 | 50<br>..... | 10<br>..... | 0<br>..... | 150 |
| 22 | 2<br>62 | 12<br>..... | .....<br>..... | 22 | 64 | 24<br>..... | 4<br>..... | .....<br>..... | 64 | 151 | .....<br>51 | .....<br>11 | .....<br>1 | 151 |
| 23 | 3<br>63 | 13<br>..... | .....<br>..... | 23 | 70 | 30<br>..... | .....<br>..... | .....<br>..... | 70 | 152 | .....<br>52 | .....<br>12 | .....<br>2 | 152 |
| 24 | 4<br>64 | 14<br>..... | .....<br>..... | 24 | 71 | .....<br>31 | .....<br>..... | .....<br>..... | 71 | 153 | .....<br>53 | .....<br>13 | .....<br>3 | 153 |
| 30 | 70.<br>– 90 | .....<br>..... | .....<br>..... | 30 | 72 | .....<br>32 | .....<br>..... | .....<br>..... | 72 | 154 | .....<br>54 | .....<br>14 | .....<br>4 | 154 |
| 31 | 71<br>101 | .....<br>..... | .....<br>..... | 31 | 73 | .....<br>33 | .....<br>..... | .....<br>..... | 73 | 161 | 61<br>..... | 21<br>..... | 1<br>..... | 161 |
| 32 | 72<br>102 | .....<br>..... | .....<br>..... | 32 | 74 | .....<br>34 | .....<br>..... | .....<br>..... | 74 | 162 | 62<br>..... | 22<br>..... | 2<br>..... | 162 |
| 33 | 73<br>103 | .....<br>..... | .....<br>..... | 33 | 81 | 41<br>..... | .....<br>..... | .....<br>..... | 81 | 163 | 63<br>..... | 23<br>..... | 3<br>..... | 163 |
| 34 | 74<br>104 | .....<br>..... | .....<br>..... | 34 | 82 | 42<br>..... | .....<br>..... | .....<br>..... | 82 | 164 | 64<br>..... | 24<br>..... | 4<br>..... | 164 |
| 41 | 91<br>81 | .....<br>..... | .....<br>..... | 41 | 83 | 43<br>..... | .....<br>..... | .....<br>..... | 83 | 170 | 70<br>..... | 30<br>..... | .....<br>..... | 170 |
| 42 | 92<br>82 | .....<br>..... | .....<br>..... | 42 | 84 | 44<br>..... | .....<br>..... | .....<br>..... | 84 | 171 | .....<br>71 | .....<br>31 | .....<br>..... | 171 |

TABLE II. (Continued.)

| | 10 | 50 | 150 | |
|---|---|---|---|---|
| 172 | 72 | 32 | | 172 |
| 173 | 73 | 33 | | 173 |
| 174 | 74 | 34 | | 174 |
| 181 | 81 | 41 | | 181 |
| 182 | 82 | 42 | | 182 |
| 183 | 83 | 43 | | 183 |
| 184 | 84 | 44 | | 184 |
| 190 | 90 | −30 | | 190 |
| 191 | 91 | 41 | | 191 |
| 192 | 92 | 42 | | 192 |
| 193 | 93 | 43 | | 193 |
| 194 | 94 | 44 | | 194 |
| 201 | 101 | 31 | | 201 |
| 202 | 102 | 32 | | 202 |
| 203 | 103 | 33 | | 203 |
| 204 | 104 | 34 | | 204 |
| 210 | 110 | | | 210 |
| 211 | 111 | | | 211 |

| | 10 | 50 | 150 | |
|---|---|---|---|---|
| 212 | 112 | | | 212 |
| 213 | 113 | | | 213 |
| 214 | 114 | | | 214 |
| 221 | 121 | | | 221 |
| 222 | 122 | | | 222 |
| 223 | 123 | | | 223 |
| 224 | 124 | | | 224 |
| 230 | −110 | | | 230 |
| 231 | 121 | | | 231 |
| 232 | 122 | | | 232 |
| 233 | 123 | | | 233 |
| 234 | 124 | | | 234 |
| 241 | 111 | | | 241 |
| 242 | 112 | | | 242 |
| 243 | 113 | | | 243 |
| 244 | 114 | | | 244 |
| 270 | 130 | | | 270 |
| 271 | 131 | | | 271 |

| | 10 | 50 | 150 | |
|---|---|---|---|---|
| 272 | 132 | | | 272 |
| 273 | 133 | | | 273 |
| 274 | 134 | | | 274 |
| 281 | 141 | | | 281 |
| 282 | 142 | | | 282 |
| 283 | 143 | | | 283 |
| 284 | 144 | | | 284 |
| 290 | −130 | | | 290 |
| 291 | 141 | | | 291 |
| 292 | 142 | | | 292 |
| 293 | 143 | | | 293 |
| 294 | 144 | | | 294 |
| 301 | 131 | | | 301 |
| 302 | 132 | | | 302 |
| 303 | 133 | | | 303 |
| 304 | 134 | | | 304 |

The following examples will show the use of the preceding Table, in forming the equations of condition which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector and of the longitude.

$$-\frac{d^2 . r^3 \delta}{dt^2}\frac{1}{r} - \mu\delta . \frac{1}{r} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

$$r^3 = a^3\left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right) - 3e\left(1 + \frac{3}{8}e^2\right)\cos(nt + \varepsilon - \varpi) + \frac{e^3}{8}\cos(3nt + 3\varepsilon - 3\varpi)\right.$$

$$\frac{(n-n_1)^2}{n^2}\left\{(1 + 3e^2)r_1 - \frac{3e^2}{2}(r_{11} + r_{21})\right\} - r_1 + \frac{m_1}{a}q_1 = 0$$

$$\frac{4(n-n_1)^2}{n^2}\left\{(1 + 3e^2)r_2 - \frac{3}{2}e^2(r_{12} + r_{22})\right\} - r_2 + \frac{m_1}{a}q_2 = 0$$

$$\frac{d\lambda}{dt} = \frac{h}{r^2} + \frac{2h}{r}\delta \cdot \frac{1}{r} - \frac{1}{r^2}\int\frac{dR}{d\lambda}dt$$

$$\frac{a^2}{r^2} = 1 + \frac{e^2}{2} + 2e\left(1 + \frac{3e^2}{8}\right)\cos(nt + \varepsilon - \varpi) + \frac{5e^2}{2}\cos(2nt + 2\varepsilon - 2\varpi)$$

$$+ \frac{13}{4}e^3\cos(3nt + 3\varepsilon - 3\varpi)$$

$$\frac{a}{r} = 1 + e\left(1 - \frac{e^2}{8}\right)\cos(nt + \varepsilon - \varpi) + e^2\cos(2nt + 2\varepsilon - 2\varpi) + \frac{9}{8}e^3\cos(3nt + 3\varepsilon - 3\varpi)$$

$$\lambda = n\{1 + 2r_0\}t + \varepsilon$$

$$+\left\{2\left\{r_1 + \frac{e^2}{2}(r_{11} + r_{21})\right\}\right.$$

$$\left.- \frac{m_1}{\mu}\left\{\left(1 + \frac{e^2}{2}\right)\frac{an\,R_1}{(n-n_1)} + \frac{e^2}{n_1}an\,R_{11} + \frac{e^2\,an\,R_{21}}{(2n-n_1)}\right\}\right\}\frac{n}{(n-n_1)}\sin(nt - n_1t + \varepsilon - \varepsilon_1)$$

$$+\left\{2\left\{r_2 + \frac{e^2}{2}(r_{12} + r_{22})\right\}\right.$$

$$\left.- \frac{m_1}{\mu}\left\{\left(1 + \frac{e^2}{2}\right)\frac{an\,R_2}{(n-n_1)} + \frac{2e^2an\,R_{12}}{(n-2n_1)} + \frac{2e^2an\,R_{22}}{(3n-2n_1)}\right\}\right\}\frac{n}{2(n-n_1)}\sin(2nt - 2n_1t + \varepsilon - \varepsilon_1)$$

In the same way, by means of the Table, all the other coefficients may be found.

The great inequality of Jupiter consists of the arguments 155, 174, 213, 273, and 312, the variable part of which is $2n - 5n_1$, and arises, as is well known, from the introduction of the square of this quantity, which is small, by successive integrations in the denominators of the coefficients of the sines in the expression for the longitude, of which the above named are the arguments.

The following are the equations which have reference to these arguments, and which may be found at once by Table II.

$$\frac{(2n-5n_1)^2}{n^2}\left\{r_{155} - \frac{3}{2}r_{54} + \frac{1}{16}r_4\right\} - r_{155} + \frac{m_1a}{\mu}q_{155} = 0$$

$$\frac{(2n-5n_1)^2}{n^2}\left\{r_{174} - \frac{3}{2}r_{74}\right\} - r_{174} + \frac{m_1a}{\mu}q_{174} = 0$$

$$\frac{(2n-5n_1)^2}{n^2}\left\{r_{213} - \frac{3}{2}r_{113}\right\} - r_{213} + \frac{m_1a}{\mu}q_{214} = 0$$

$$\frac{(2n-5n_1)^2}{n^2}\left\{r_{373}-\frac{3}{2}r_{133}\right\}-r_{373}+\frac{m_1 a}{\mu}q_{373}=0$$

$$\frac{(2n-5n_1)^2}{n^2}\left\{r_{313}-r_{313}\right\}+\frac{m_1 a}{\mu}q_{313}=0$$

$$\delta\lambda=\left\{2\left\{r_{155}+\frac{1}{2}\left(r_{55}+r_{15}+\frac{9}{8}r_4\right)\right\}\right.$$

$$\left.-\frac{m_1}{\mu}\left\{\frac{5na}{(2n-5n_1)}R_{155}+\frac{5na\,R_{55}}{(3n-5n_1)}+\frac{5.5na\,R_{15}}{4\,(3n-4n_1)}+\frac{13.5na\,R_5}{8.5\,(n-n_1)}\right\}\right\}\frac{ne^3}{(2n-5n_1)}\sin(2nt-5n_1t+3\varpi)$$

$$+\left\{2\left\{r_{174}+\frac{1}{2}(r_{74}+r_{34})\right\}\right.$$

$$\left.-\frac{m_1}{\mu}\left\{\frac{4na\,R_{174}}{(2n-5n_1)}+\frac{4na\,R_{74}}{(3n-5n_1)}+\frac{5.4.na\,R_{34}}{4\,(4n-5n_1)}\right\}\right\}\frac{ne^2e_1}{(2n-5n_1)}\sin(2nt-5n_1t+2\varpi+\varpi_1)$$

$$+\left\{2\left\{r_{213}+\frac{1}{2}r_{113}\right\}-\frac{m_1}{\mu}\left\{\frac{3na\,R_{213}}{(2n-5n_1)}+\frac{3na\,R_{113}}{(3n-5n_1)}\right\}\right\}\frac{nee_1^2}{(2n-5n_1)}\sin(2nt-5n_1t+\varpi+2\varpi_1)$$

$$+\left\{2\left\{r_{373}+\frac{1}{2}r_{133}\right\}-\frac{m_1}{\mu}\left\{\frac{3na\,R_{273}}{(2n-5n_1)}+\frac{3na\,R_{153}}{(3n-5n_1)}\right\}\right\}\frac{ne\sin^2\frac{i_1}{2}}{(2n-5n_1)}\sin(2nt-5n_1t+\varpi+2\nu_1)$$

$$+\left\{2r_{313}-\frac{2m_1na\,R_{313}}{\mu(2n-5n_1)}\right\}\frac{ne_1\sin^2\frac{i_1}{2}}{(2n-5n_1)}\sin(2nt-5n_1t+\varpi_1+2\nu_1)$$

The quantities $r_{55}$, $r_{74}$, $r_{113}$ and $r_{133}$ have the quantity $2n-5n_1$ in the denominator, rejecting those quantities in the value of $\delta\lambda$ which have not $(2n-5n_1)^2$ in the denominator.

$$r_{155}=-\frac{4m_1n^3a\,R_{155}e^3}{\mu\,(n-5n_1)(3n-5n_1)(2n-5n_1)}$$

$$r_{174}=-\frac{4m_1n^3a\,R_{174}e^2e_1}{\mu\,(n-5n_1)(3n-5n_1)(2n-5n_1)}$$

$$r_{313}=-\frac{4m_1n^3a\,R_{313}ee_1^2}{\mu\,(n-5n_1)(3n-5n_1)(2n-5n_1)}$$

$$r_{373}=-\frac{4m_1n^3a\,R_{373}e\sin^2\frac{i_1}{2}}{\mu\,(n-5n_1)(3n-5n_1)(2n-5n_1)}$$

$$r_{313}=-\frac{4m_1n^3a\,R_{313}e_1\sin^2\frac{i_1}{2}}{\mu\,(n-5n_1)(3n-5n_1)(2n-5n_1)}$$

$$\delta\lambda=\left\{2r_{155}+r_{55}-\frac{5m_1na\,R_{155}}{\mu(2n-5n_1)}\right\}\frac{ne^3}{(2n-5n_1)}\sin(2nt-5n_1t+3\varpi)$$

$$+\left\{2r_{174}+r_{74}-\frac{4m_1na\,R_{174}}{\mu(2n-5n_1)}\right\}\frac{ne^2e_1}{(2n-5n_1)}\sin(2nt-5n_1t+2\varpi+\varpi_1)$$

$$+\left\{2r_{313}+r_{133}-\frac{3m_{,}an\,R_{313}}{\mu(2n-5n_{,})}\right\}\frac{nee_{,}^{2}}{(2n-5n_{,})}\sin(2nt-5n_{,}t+\varpi+2\varpi_{,})$$

$$+\left\{2r_{373}+r_{139}-\frac{3m_{,}an\,R_{373}}{\mu(2n-5n_{,})}\right\}\frac{ne\sin^{2}\frac{i_{,}}{2}}{(2n-5n_{,})}\sin(2nt-5n_{,}t+\varpi+2\nu_{,})$$

$$+\left\{2r_{313}-\frac{2m_{,}an\,R_{313}}{\mu(2n-5n_{,})}\right\}\frac{ne_{,}\sin^{2}\frac{i_{,}}{2}}{(2n-5n_{,})}\sin(2nt-5n_{,}t+\varpi_{,}+2\nu_{,})$$

The coefficients of the terms in the development of $R$ multiplied by the cubes of the eccentricities, as regards the quantities $b_5$ and $b_7$, (they also contain the quantities $b_{33}$) may be found by changing $b_3$ into $b_5$, in the terms in $R$ multiplied by the eccentricities, and multiplying the result by

$$-\frac{9}{8}\frac{(a^{2}e^{2}+a_{,}^{2}e_{,}^{2})}{a_{,}^{2}}+\frac{3}{8}\frac{a^{2}}{a_{,}^{2}}e^{2}\cos 2x-\frac{3}{4}\frac{a}{a_{,}}\left(e^{2}+e_{,}^{2}+2\sin^{2}\frac{i_{,}}{2}\right)\cos t+\frac{9}{16}\frac{a}{a_{,}}e^{2}\cos(t+2x)$$
[0] [50] [1] [61]

$$-\frac{9}{16}\frac{a}{a_{,}}e_{,}^{2}\cos(t-2z)+\frac{3}{16}\frac{a}{a_{,}}e^{2}\cos(t-2x)+\frac{3}{4}\frac{a}{a_{,}}e_{,}^{2}\cos(t+2z)+\frac{27}{8}\frac{a}{a_{,}}ee_{,}\cos(t-x+z)$$
[111] [51] [121] [91]

$$-\frac{9}{8}\frac{a}{a_{,}}ee_{,}\cos(t+x+z)-\frac{9}{8}\frac{a}{a_{,}}ee_{,}\cos(t-x-z)+\frac{3}{8}\frac{a}{a_{,}}ee_{,}\cos(t+x-z)$$
[81] [71] [101]

$$+\frac{3}{2}\frac{a}{a_{,}}\sin^{2}\frac{i_{,}}{2}\cos(t+2y)+\frac{3}{8}e_{,}^{2}\cos 2z$$
[141] [110]

and changing $b_5$ into $b_7$, in the terms in $R$ multiplied by the squares and products of the eccentricities, and multiplying the result by

$$-\frac{5}{6}\text{ and }-\frac{2a^{2}}{a_{,}^{2}}e\cos x+\frac{3a}{a_{,}}e\cos(t-x)+\frac{3a}{a_{,}}e_{,}\cos(t+z)-\frac{a}{a_{,}}e\cos(t+x)$$
[10] [11] [41] [21]

$$-\frac{a}{a_{,}}e_{,}\cos(t-z)-2e_{,}\cos z$$
[31] [30]

and changing $b_3$ into $b_5$ in the terms in $R$ multiplied by the squares and products of the eccentricities, and multiplying the result by $-\frac{3}{4}$ and the same quantity.

Thus $R_{155}$ results from the combination of the arguments

51 × 14, 50 × 15, 61 × 16, 10 × 55, and 11 × 54.

51 × 14 gives $+\frac{3}{32}\frac{a}{a_{,}}\left\{\frac{3a}{4a_{,}^{2}}b_{5,3}-\frac{a^{2}}{2a_{,}^{3}}b_{5,4}-\frac{a}{4a_{,}^{4}}b_{5,5}\right\}$

$50 \times 15$ gives $+\frac{3}{16}\frac{a^2}{a_1^2}\left\{\frac{3a}{4a_1^3}b_{5,4}-\frac{a^2}{2a_1^4}b_{5,5}-\frac{a}{4a_1^3}b_{5,6}\right\}$.

$61 \times 16$ gives $+\frac{9}{32}\frac{a}{a_1}\left\{\frac{3a}{4a_1^3}b_{5,5}-\frac{a^2}{2a_1^4}b_{5,6}-\frac{a}{4a_1^3}b_{5,7}\right\}$

$$R_{55}=-\frac{a}{16a_1^2}b_{5,4}-\frac{a^2}{8a_1^3}b_{5,5}-\frac{3a}{16a_1^3}b_{5,6}-\frac{3.9}{2.4.4}\frac{a^2}{a_1^3}b_{5,5}+\frac{3.3}{2.4}\frac{a^3}{a_1^4}b_{5,4}$$

$$-\frac{3a^2}{2.4.2}\frac{(2a^2-3a_1^2)}{a_1^5}b_{5,5}-\frac{3}{2.4}\frac{a^3}{a_1^4}b_{5,6}-\frac{3a^2}{2.4.4a_1^3}b_{5,7}$$

changing $b_3$ into $-\frac{3}{4}b_{5,5}$ and $b_5$ into $-\frac{5}{6}b_{7,7}$ we have

$$\frac{3a}{64a_1^2}b_{5,4}+\frac{3a^2}{32a_1^3}b_{5,5}+\frac{9}{64}\frac{a}{a_1^2}b_{5,6}+\frac{3.9.5}{2.4.4.6}\frac{a^2}{a_1^3}b_{7,5}-\frac{3.3.5}{2.4.6}\frac{a^3}{a_1^4}b_{7,4}$$

$$+\frac{3.5.a^2}{2.4.2}\frac{(2a^2-3a_1^2)}{a^5}b_{7,5}+\frac{3.5}{2.4.6}\frac{a^3}{a_1^4}b_{7,6}+\frac{3.5}{2.4.4.6}\frac{a^2}{a_1^3}b_{7,7}$$

$$=\frac{3}{64}\frac{a}{a_1^2}b_{5,4}+\frac{3}{32}\frac{a^2}{a_1^3}b_{5,5}+\frac{9}{64}\frac{a}{a_1^2}b_{5,6}+\frac{3.5}{8.6}\frac{a^3}{a_1^5}\left\{\frac{a^2+a_1^2}{a_1^2}b_{7,5}-\frac{a}{a_1}b_{7,4}-\frac{a}{a_1}b_{7,6}\right\}$$

$$+\frac{3.9.5}{8.4.6}\frac{a^2}{a_1^3}\left\{b_{7,5}-b_{7,5}\right\}-\frac{3.5}{4.6}\frac{a^3}{a_1^4}\left\{b_{7,4}-b_{7,6}\right\}-\frac{3.5}{32.6}\frac{a^2}{a_1^3}\left\{b_{7,5}-b_{7,7}\right\}$$

and since $b_{5,5}=\frac{a^2+a_1^2}{a_1^3}b_{5,5}-\frac{a}{a_1}b_{7,4}-\frac{a}{a_1}b_{7,6}$

$$4b_{5,4}=\frac{5}{2}\frac{a}{a_1}\left\{b_{7,5}-b_{7,5}\right\}\quad 5b_{5,5}=\frac{5}{2}\frac{a}{a_1}\left\{b_{7,4}-b_{7,6}\right\}\quad 6b_{5,5}=\frac{5}{2}\frac{a}{a_1}\left\{b_{7,5}-b_{7,7}\right\}$$

$$=\frac{3}{64}\frac{a}{a_1^2}b_{5,4}+\frac{3}{32}\frac{a^2}{a_1^3}b_{5,5}+\frac{9}{64}\frac{a}{a_1^2}b_{5,6}+\frac{15}{48}\frac{a^2}{a_1^3}b_{5,5}+\frac{27}{24}\frac{a}{a_1^2}b_{5,4}-\frac{15}{12}\frac{a^2}{a_1^3}b_{5,5}-\frac{3}{16}\frac{a}{a_1^2}b_{5,6}$$

$$=\frac{75}{64}\frac{a}{a_1^2}b_{5,4}-\frac{27}{32}\frac{a^2}{a_1^3}b_{5,5}-\frac{3}{64}\frac{a}{a_1^2}b_{5,6}$$

$$R_{54}=-\frac{a}{16a_1^2}b_{5,5}-\frac{a^2}{8a_1^3}b_{5,4}-\frac{3a}{16a_1^3}b_{5,5}-\frac{3.9a^2}{2.4.4a_1^3}b_{5,5}+\frac{3.3a^3}{2.4a_1^4}b_{5,5}$$

$$-\frac{3a^3(2a^2-3a_1^2)}{2.4.2a_1^5}b_{5,4}-\frac{3a^3}{2.4a_1^4}b_{5,6}-\frac{3a^2}{2.4.4a_1^3}b_{5,6}$$

Similar changes and reductions give

$$\frac{57a}{64a_1^2}b_{5,5}-\frac{19a^2}{32a_1^3}b_{5,4}-\frac{a}{64a_1^2}b_{5,5}$$

$$R_{155} = \frac{3}{32}\frac{a}{a_i}\left\{\frac{3}{4}\frac{a}{a_i^2}b_{5,5} - \frac{a^2}{2a_i^3}b_{5,4} - \frac{a}{4a_i^2}b_{5,5}\right\} + \frac{3}{16}\frac{a^2}{a_i^3}\left\{\frac{3}{4}\frac{a}{a_i^2}b_{5,4} - \frac{a^2}{2a_i^3}b_{5,5} - \frac{a}{4a_i^2}b_{5,6}\right\}$$
$$+ \frac{9a}{32a_i}\left\{\frac{3a}{4a_i^2}b_{5,4} - \frac{a^2}{2a_i^3}b_{5,5} - \frac{a}{4a_i^2}b_{5,7}\right\} - \frac{a^2}{a_i^3}\left\{\frac{75}{64}\frac{a}{a_i^2}b_{5,4} - \frac{27}{32}\frac{a^2}{a_i^3}b_{5,5} - \frac{3}{64}\frac{a}{a_i^2}b_{5,6}\right\}$$
$$+ \frac{3a}{2a_i}\left\{\frac{57}{64}\frac{a}{a_i^2}b_{5,5} - \frac{19}{32}\frac{a^2}{a_i^3}b_{5,4} - \frac{a}{64a_i^2}b_{5,5}\right\}$$

and adding the terms which depend upon $b_3$,

$$R_{155} = \frac{a}{96a_i^2}b_{5,4} - \frac{a^2}{16a_i^3}b_{5,5} + \frac{a}{12a_i^2}b_{5,6} + \frac{45}{32}\frac{a^2}{a_i^3}b_{5,5} - \frac{63}{32}\frac{a^3}{a_i^4}b_{5,4} + \frac{(21a_i^2 + 96a^2)}{128a_i^5}a^2b_{5,5}$$
$$- \frac{9}{64}\frac{a^3}{a_i^4}b_{5,6} - \frac{9}{128}\frac{a^3}{a_i^5}b_{5,7}$$

which may be still further reduced. $R_{174}$, $R_{213}$, $R_{273}$, and $R_{312}$ may be obtained in a similar manner.

The following Table shows the arguments which, by their combination with the arguments 1, 2, 3, 12, 13, 31, 32, 64, 65, 73, 74, 112, and 113, by addition and subtraction produce the arguments 155, 174, 213, 273, and 312.

| | 1 | 2 | 3 | 12 | 13 | 31 | 32 | 64 | 65 | 73 | 74 | 112 | 113 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 155 | 154 | 153 | 152 | 53 | 52 | ..... | ..... | 11 | ..... | 192 | 191 | ..... | ..... | 155 |
| | 156 | 157 | 158 | ..... | ..... | ..... | ..... | ..... | − 10 | ..... | ..... | ..... | ..... | |
| 174 | 173 | 172 | 171 | 72 | 71 | 53 | 52 | ..... | ..... | 11 | ..... | 192 | 191 | 174 |
| | 175 | 176 | 177 | ..... | ..... | ..... | ..... | − 30 | − 41 | ..... | − 10 | ..... | ..... | |
| 213 | 212 | 211 | −210. | 111 | ..... | 72 | 71 | ..... | ..... | ..... | ..... | 11 | ..... | 213 |
| | 214 | 215 | 216 | ..... | −110 | ..... | ..... | −231 | −232 | − 30 | − 41 | ..... | − 10 | |
| 273 | 272 | 271 | −270. | 131 | ..... | ..... | ..... | ..... | ..... | 330 | ..... | ..... | ..... | 273 |
| | 274 | 275 | 276 | ..... | −130 | ..... | ..... | −291 | −292 | ..... | −331 | ..... | ..... | |
| 312 | 311 | −310. | −321. | ..... | ..... | 131 | ..... | ..... | ..... | ..... | ..... | 330 | ..... | 312 |
| | 313 | 314 | 315 | ..... | ..... | ..... | −130 | ..... | ..... | −291 | −292 | ..... | −331 | |

If

$$r\,\delta.\frac{1}{r} = r'_1\cos(nt - n_it) + r'_2\cos(2nt - 2n_it) + r'_3\cos(3nt - 3n_it) + er'_{12}\cos(nt - 2n_it + \varpi)$$
$$+ er'_{13}\cos(2nt - 3n_it + \varpi) + \&c.$$

$$r_i\,\delta.\frac{1}{r_i} = r'_{i1}\cos(nt - n_it) + r'_{i2}\cos(2nt - 2n_it) + r'_{i3}\cos(3nt - 3n_it) + er'_{i12}\cos(nt - 2n_it + \varpi)$$
$$+ er'_{i13}\cos(2nt - 3n_it + \varpi) + \&c.$$

$$\delta\lambda = \lambda_1\sin(nt - n_it) + \lambda_2\sin(2nt - 2n_it) + \lambda_3\sin(3nt - 3n_it) + e\lambda_{12}\sin(nt - 2n_it + \varpi)$$
$$+ e\lambda_{13}\sin(2nt - 3n_it + \varpi) + \&c.$$

$$\delta\lambda_i = \lambda_{i1}\sin(nt - n_i t) + \lambda_{i2}\sin(2nt - 2n_i t) + \lambda_{i3}\sin(3nt - 3n_i t) + e\lambda_{i\,12}\sin(nt - 2n_i t + \varpi) + e\lambda_{i\,13}\sin(2nt - 3n_i t + \varpi) + \&c.$$

Supposing that the arguments 1, 2, 3, 12, 13, 31, 32, 64, 65, 73, 74, 112, 113, 155, 174, 213, 273, and 312 are alone sensible in $\delta\,.\,\frac{1}{r}$, $\delta\lambda$, $\delta\frac{1}{r_i}$ and $\delta\lambda_i$ the coefficient of $e^3\cos(2nt - 5n_i t + 3\varpi)$ in the expression for $\delta R$ or $\delta R_{155}$

$$= -\frac{1}{2}\left\{\frac{a\,\mathrm{d}\,.\,R_{154}}{\mathrm{d}a} + \frac{a\,\mathrm{d}\,.\,R_{156}}{\mathrm{d}a}\right\}r'_1 + \left\{2R_{154} - 3R_{156}\right\}\left\{\lambda_1 - \lambda_{j1}\right\} - \frac{1}{2}\left\{\frac{a\,\mathrm{d}\,R_{153}}{\mathrm{d}a} + \frac{a\,\mathrm{d}\,R_{157}}{\mathrm{d}a}\right\}r'_2$$

$$+ \frac{1}{2}\left\{3R_{155} - 7R_{157}\right\}\left\{\lambda_2 - \lambda_{j2}\right\} - \frac{1}{2}\left\{\frac{a\,\mathrm{d}\,.\,R_{156}}{\mathrm{d}a} + \frac{a\,\mathrm{d}\,.\,R_{158}}{\mathrm{d}a}\right\}r'_3$$

$$+ \left\{R_{159} - 4R_{158}\right\}\left\{\lambda_3 - \lambda_{j3}\right\} - \frac{a\,\mathrm{d}\,.\,R_{33}}{2\,\mathrm{d}\,a}r'_{12} + \frac{3}{2}R_{33}\left\{\lambda_{12} - \lambda_{j12}\right\}$$

$$- \frac{a\,\mathrm{d}\,.\,R_{39}}{2\,\mathrm{d}\,a}r'_{13} + R_{39}\left\{\lambda_{13} - \lambda_{j13}\right\} - \frac{a\,\mathrm{d}\,.\,R_{64}}{2\,\mathrm{d}\,a}r'_{11} + 2R_{64}\left\{\lambda_{11} - \lambda_{j11}\right\}$$

$$- \frac{a\,\mathrm{d}\,.\,R_{65}}{2\,\mathrm{d}\,a}r'_{10} - \frac{5}{2}R_{65}\left\{\lambda_{10} - \lambda_{j10}\right\} - \frac{a\,\mathrm{d}\,R_{192}}{2\,\mathrm{d}\,a}r'_{73} - R_{192}\left\{\lambda_{73} - \lambda_{j73}\right\} - \frac{a\,\mathrm{d}\,R_{193}}{2\,\mathrm{d}\,a}r'_{74}$$

$$- \frac{1}{2}R_{193}\left\{\lambda_{74} - \lambda_{j74}\right\} - \frac{a\,\mathrm{d}\,.\,R_{5}}{\mathrm{d}\,a}r'_{155} - \frac{1}{2}\left\{\frac{a_j\,\mathrm{d}\,.\,R_{154}}{\mathrm{d}\,a_j} + \frac{a_j\,\mathrm{d}\,.\,R_{156}}{\mathrm{d}\,a_j}\right\}r'_{j1}$$

$$- \frac{1}{2}\left\{\frac{a_j\,\mathrm{d}\,.\,R_{155}}{\mathrm{d}\,a_j} + \frac{a_j\,\mathrm{d}\,.\,R_{157}}{\mathrm{d}\,a_j}\right\}r'_{j2} - \frac{1}{2}\left\{\frac{a_j\,\mathrm{d}\,R_{156}}{\mathrm{d}\,a_j} + \frac{a_j\,\mathrm{d}\,R_{158}}{\mathrm{d}\,a_j}\right\}r'_{j3} - \frac{a_j\,\mathrm{d}\,.\,R_{33}}{2\,\mathrm{d}\,a_j}r'_{j\,10}$$

$$- \frac{a_j\,\mathrm{d}\,.\,R_{39}}{2\,\mathrm{d}\,a_j}r'_{j\,13} - \frac{a_j\,\mathrm{d}\,.\,R_{64}}{2\,\mathrm{d}\,a_j}r'_{j\,11} - \frac{a_j\,\mathrm{d}\,.\,R_{65}}{2\,\mathrm{d}\,a_j}r'_{j\,10} - \frac{a_j\,\mathrm{d}\,R_{192}}{2\,\mathrm{d}\,a_j}r_{j73} - \frac{a_j\,\mathrm{d}\,R_{193}}{2\,\mathrm{d}\,a_j}r_{j74} - \frac{a_j\,\mathrm{d}\,.\,R_{5}}{\mathrm{d}\,a_j}r'_{j\,155}$$

In the same way the expression for $\delta\,.\,R_{174}$, $\delta\,.\,R_{213}$, $\delta\,.\,R_{273}$, and $\delta\,.\,R_{312}$ may be found from the preceding Table.

If $a < a_j$ and

$$\left\{1 - \frac{a}{a_j}\cos\theta + \frac{a^2}{a_j^2}\right\}^{-\frac{1}{2}} = \tfrac{1}{2}b_{1,0}^{\,*} + b_{1,1}\cos\theta + b_{1,2}\cos 2\theta\ \&c.$$

$$\left\{1 - \frac{a}{a_j}\cos\theta + \frac{a^2}{a_j^2}\right\}^{-\frac{3}{2}} = \tfrac{1}{2}b_{3,0}^{\,*} + b_{3,1}\cos\theta + b_{3,2}\cos 2\theta\ \&c.$$

$$R = m_i\left\{\frac{a}{a_j^2}\left(\cos^2\tfrac{i_j}{2} - \frac{e^2 + e_j^2}{2}\right)\cos(nt - n_i t)\right.\dagger$$

$$- \frac{3m_j}{2}\frac{a}{a_j^2}e\cos(n_i t - \varpi) + \frac{m_j a}{a_j^2}e\cos(2nt - n_i t - \varpi) + \frac{2m_j a}{2a_j^2}e_j\cos(nt - 2n_i t + \varpi_j)$$

* The notation is slightly changed from that used before.

† $s$ and $s_j$ which accompany $nt$ and $n_j t$ are omitted for convenience.

General expression for the development of $R$.

$$+\frac{m_i a}{8 a_i^2} e^2 \cos(n t + n_i t - 2\varpi) + \frac{3 m_i a}{8 a_i^2} e^2 \cos(3 n t - n_i t - 2\varpi) - \frac{3 m_i a}{a_i^2} e e_i \cos(2 n_i t - \varpi - \varpi_i)$$

$$+\frac{m_i a}{a_i^2} e e_i \cos(2 n t - 2 n_i t - \varpi + \varpi_i) + \frac{27}{8}\frac{m_i a}{a_i^2} e_i^2 \cos(n t - 3 n_i t + 2\varpi_i)$$

$$+\frac{m_i a}{8 a_i^2} e_i^2 \cos(n t + n_i t - 2\varpi_i) + \frac{m_i a}{a_i^2} \sin^2\frac{i_i}{2} \cos(n t + n_i t - 2\nu_i)$$

$$+ m_i \Sigma \left\{ -\frac{b_{1,i}}{2 a_i} + \frac{a}{4 a_i^2} \sin^2\frac{i_i}{2} \left( b_{3,i-1} + b_{3,i+1} \right) \right.$$

$$\left. + \frac{a (e^2 + e_i^2)}{16 a_i^2} \left( (3i - 1) b_{3,i-1} - (3i + 1) b_{3,i+1} \right) \right\} \cos i (n t - n_i t)$$

$$+ m_i \Sigma \left\{ -\frac{a}{4 a_i^2} b_{3,i-1} - \frac{a^2}{2 a_i^3} b_{3,i} + \frac{3 a}{4 a_i^2} b_{3,i+1} \right\} e \cos\left( i (n t - n_i t) + n t - \varpi \right)$$

$$+ m_i \Sigma \left\{ \frac{3}{4}\frac{a}{a_i^2} b_{3,i-1} - \frac{1}{2 a_i} b_{3,i} - \frac{a}{4 a_i^2} b_{3,i+1} \right\} e_i \cos\left( i (n t - n_i t) + n_i t - \varpi_i \right)$$

$$+ m_i \Sigma \left\{ -\frac{(2+i)}{16}\frac{a}{a_i^2} b_{3,i-1} - \frac{(1+i)}{2}\frac{a^2}{a_i^3} b_{3,i} \right.$$

$$\left. + \frac{(8+9i)}{16}\frac{a}{a_i^2} b_{3,i+1} \right\} e^2 \cos\left( i (n t - n_i t) + 2 n t - 2\varpi \right)$$

$$+ m_i \Sigma \left\{ \frac{(3+9i)}{8}\frac{a}{a_i^2} b_{3,i-1} - \frac{i}{a_i} b_{3,i} \right.$$

$$\left. - \frac{(1+i)}{8}\frac{a}{a_i^2} b_{3,i+1} \right\} e e_i \cos\left( i (n t - n_i t) + n t + n_i t - \varpi - \varpi_i \right)$$

$$+ m_i \Sigma \left\{ -\frac{(1+3i)}{8}\frac{a}{a_i^2} b_{3,i-1} \right.$$

$$\left. + \frac{3(1+i)}{8}\frac{a}{a_i^2} b_{3,i+1} \right\} e e_i \cos\left( i (n t - n_i t) + n t - n_i t - \varpi + \varpi_i \right)$$

$$+ m_i \Sigma \left\{ \frac{(8-9i)}{16}\frac{a}{a_i^2} b_{3,i-1} + \frac{(1-i)}{2 a_i} b_{3,i} \right.$$

$$\left. - \frac{(2-i)}{16}\frac{a}{a_i^2} b_{3,i+1} \right\} e_i^2 \cos\left( i (n t - n_i t) + 2 n_i t - 2\varpi_i \right)$$

$$- m_i \Sigma \frac{a}{2 a_i^2} b_{3,i-1} \sin^2\frac{i_i}{2} \cos\left( i (n t - n_i t) + 2 n_i t - 2\nu_i \right)$$

$i$ being every whole number, positive and negative and zero, and observing that $b_{m,n} = b_{m,-n}$. Considering only the terms multiplied by $e$ and $e_i$,

$$r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) = -\frac{3 m_i}{2}\frac{a}{a_i^2} e \cos(n_i t - \varpi) + \frac{m_i a}{2 a_i^2} e \cos(2 n t - n_i t - \varpi)$$

$$+\frac{m_1 a}{2 a_1^2} e_1 \cos(n t - 2 n_1 t + \varpi_1)$$

$$+ m_1 \Sigma \left\{ -\frac{i}{4}\frac{a}{a_1^2} b_{3,i-1} + \frac{(1+2i)}{2}\frac{a^2}{a_1^3} b_{3,i} - \frac{3i}{4}\frac{a}{a_1^2} b_{3,i+1} \right\} e \cos\left(i(n t - n_1 t) + n t - \varpi\right)$$

$$+ m_1 \Sigma \left\{ -\frac{3(1+i)}{4}\frac{a}{a_1^2} b_{3,i-1} + \frac{ia}{a_1} b_{3,i} + \frac{(1-i)}{4} b_{3,i+1} \right\} e_1 \cos\left(i(n t - n_1 t) + n_1 t - \varpi_1\right)$$

$$\frac{a}{r} = -\frac{m_1}{\mu}\frac{n^2}{(3n - n_1)(n - n_1)}\left\{\frac{2n}{2n - n_1} + \frac{1}{2}\right\}\frac{a^2}{a_1^2} e \cos(2 n t - n_1 t - \varpi)$$

$$-\frac{m_1}{\mu}\frac{3 n^2}{2(n - n_1)(n + n_1)}\frac{a^3}{a_1^3} e \cos(n_1 t - \varpi)$$

$$+\frac{m_1}{\mu}\frac{n^2}{n_1(2n - 2n_1)}\left\{\frac{2n}{(n - 2n_1)} + 1\right\}\frac{a^2}{a_1^2} e_1 \cos(n t - 2 n_1 t + \varpi_1)$$

$$+\Sigma\frac{n^2}{\left(i(n - n_1) + 2n\right) i (n - n_1)}\left\{\frac{3\left(i(n - n_1) + n\right)}{2 n^2} 2 r_1 *\right.$$

$$-\frac{m_1}{\mu}\left\{\frac{2(1+i)n}{i(n - n_1) + n}\left\{-\frac{a^2}{4 a_1^2} b_{3,i-1} - \frac{a^2}{2 a_1^2} b_{3,i} + \frac{3 a^2}{4 a_1^2} b_{3,i+1}\right\}\right.$$

$$\left.\left.-\frac{i}{4}\frac{a^2}{a_1^2} b_{3,i-1} + \frac{(1+2i)}{2}\frac{a^2}{a_1^3} b_{3,i} - \frac{3i}{4}\frac{a^2}{a_1^2} b_{3,i+1}\right\}\right\} e \cos\left(i(n t - n_1 t) + n t - \varpi\right)$$

$$+\frac{m_1}{\mu}\Sigma\frac{n^2}{(1-i)(n - n_1)\left((i+1)(n - n_1) + 2 n_1\right)}\left\{\frac{2 i n}{i(n - n_1) + n_1}\left\{\frac{3 a^2}{4 a_1^2} b_{3,i-1}\right.\right.$$

$$\left.\left.-\frac{a}{2 a_1} b_{3,i} - \frac{a^2}{4 a_1^2} b_{3,i+1}\right\} - \frac{3(1+i)}{4}\frac{a^2}{a_1^2} b_{3,i-1}\right.$$

$$\left.+\frac{ia}{a_1} b_{3,i} + \frac{(1-i)}{4}\frac{a^2}{a_1^2} b_{3,i+1}\right\} e_1 \cos\left(i(n t - n_1 t) + n_1 t - \varpi_1\right)$$

$$\lambda = -\left\{\frac{3 n^2}{2 n_1^2} + \frac{n^2}{n_1(n - n_1)}\frac{m_1}{\mu}\right\}\frac{a^2}{a_1^2} e \sin(n_1 t - \varpi)$$

$$-\left\{\frac{n^2}{(2n - n_1)^2} + \frac{n^2}{(2n - n_1)(n - n_1)}\right\}\frac{m_1}{\mu}\frac{a^2}{a_1^2} e \sin(2 n t - n_1 t - \varpi)$$

$$-\frac{2 n^2}{(n - 2 n_1)^2}\frac{m_1}{\mu}\frac{a^2}{a_1^2} e_1 \sin(n t - 2 n_1 t + \varpi_1)$$

* $r_1$ being the coefficient of $\cos\left(i(n t - n_1 t)\right)$ in the expression for $\frac{a}{r}$.

$$+\Sigma\dagger\frac{n}{i(n-n_1)+n}\left\{2\left(r^*+\frac{r_i}{2}\right)-\frac{m_1 n i}{\mu\left(i(n-n_1)+n\right)}\right\}\left(-\frac{a^2}{4a_i^2}b_{3,i-1}-\frac{a^3}{2a_i^3}b_{3,i}\right.$$

$$\left.+\frac{3a^3}{4a_i^3}b_{3,i+1}\right)+\frac{m_1 n}{\mu(n-n_1)}\frac{a}{a_1}b_{1,i}\Big\}e\sin\left(i(nt-n_1t)+nt-\varpi\right)$$

$$-\Sigma\frac{n}{i(n-n_1)+n_1}\left\{2r^*-\frac{m_1 n i}{\mu\left(i(n-n_1)+n_1\right)}\right\}\left(\frac{3}{4}\frac{a^2}{a_1^2}b_{3,i-1}-\frac{a}{2a_1}b_{3,i}\right.$$

$$\left.-\frac{a^2}{4a_1^2}b_{3,i+1}\right)e_1\sin\left(i(nt-n_1t)+n_1t-\varpi_1\right)$$

If $a > a_1$, and

$$\left\{1-\frac{a_1}{a}\cos\theta+\frac{a_1^2}{a^2}\right\}^{-\frac{1}{2}}=\tfrac{1}{2}b_{1,0}+b_{1,1}\cos\theta+b_{1,2}\cos 2\theta+\&c.$$

$$\left\{1-\frac{a_1}{a}\cos\theta+\frac{a_1^2}{a^2}\right\}^{-\frac{3}{2}}=\tfrac{1}{2}b_{3,0}+b_{3,1}\cos\theta+b_{3,2}\cos 2\theta+\&c.$$

the value of $R$ may be easily inferred from the value which it has in the former case. Considering only the terms multiplied by the eccentricities

$$r\left(\frac{dR}{dr}\right)=-\frac{3m_1}{2}\frac{a}{a_1^2}e\cos(nt-\varpi)+\frac{m_1}{2}\frac{a}{a_1^2}e\cos(2nt-n_1t-\varpi)$$

$$+\frac{m_1}{2}\frac{a}{a_1^2}e_1\cos(nt-2n_1t+\varpi_1)$$

$$+m_1\Sigma\left\{-\frac{i}{4}\frac{a_1}{a^2}b_{3,i-1}+\frac{(1+2i)}{2a}b_{3,i}\right.$$

$$\left.-\frac{3i}{4}\frac{a_1}{a^2}b_{3,i+1}\right\}e\cos\left(i(nt-n_1t)+nt-\varpi\right)$$

$$+m_1\Sigma\left\{-\frac{3(1+i)}{4}\frac{a_1}{a^2}b_{3,i-1}+\frac{ia_1^2}{a^3}b_{3,i}\right.$$

$$\left.+\frac{(1-i)}{4}\frac{a_1}{a^2}b_{3,i+1}\right\}e_1\cos\left(i(nt-n_1t)+n_1t-\varpi_1\right)$$

All these expressions are to a certain extent arbitrary, on account of the equation which connects $b_{s,i-1}$, $b_{s,i}$, and $b_{s,i+1}$

$$\frac{(2i+1)}{2}\frac{a}{a_1}b_{3,i+1}=\frac{i(a^2+a_1^2)}{a_1^2}b_{3,i}-\frac{(2i-1)}{2}\frac{a}{a_1}b_{3,i-1}$$

† $r^*$ being the coefficient of the cosine of the same argument in the expression for $\frac{a}{r}$ and excluding the case of $i=0$.

The reader is requested to make the following corrections.

Page 50, line 4, read $q_6 = -\frac{3a}{2a_i^2} + \frac{3}{2}\frac{a}{a_i^2} b_{3,0} - \frac{a^2}{2a_i^3} b_{3,1} + \frac{a}{4a_i^2} b_{3,2}$

Page 53, line 3, read $= \frac{m_i}{\mu}\left\{\frac{2a^3}{a_i^3} b_{3,0} - \frac{5}{4}\frac{a^2}{a_i^2} b_{3,1}\right\}$

Page 247, line 1, read $\lambda = n t$
$+ \lambda_1 \sin 2t$
$+ e\lambda_2 \sin x$
$+ e\lambda_3 \sin (2t - x)$
$+ e\lambda_4 \sin (2t + x)$
$+ e_i\lambda_5 \sin z$ &c. &c.

for $\lambda = n t$
$+ \lambda_1 \cos 2t$
$+ e\lambda_2 \cos x$ &c. &c.

Page 254, line 1, read $-\frac{3}{2} e^2 e_i \cos (2t + 2x + z)$
[25] [30]

Page 260, line 6, read $+\left\{3 - \frac{15}{2}\right\} e e_i \cos (x - z - 2y)$
[89]

Page 262, line 6, read $-\frac{15}{32} e e_i^2 \cos (2t + x - 3z)$
[58]

Page 265, line 1, read $+\frac{25}{64}\frac{a^3}{a_i^3} e^3 e_i \cos (2t + 3x + z) + \frac{3}{32}\frac{a^3}{a_i^3} e^3 e_i \cos (3x - z)$
[43] [44]

Page 274, line 6, read $+\left\{2r_3 + r_1 - \left\{\frac{9}{2(2 - m - c)}\right.\right.$ &c.

Page 274, line 7, read $+\left\{2r_4 + r_1 - \left\{-\frac{3}{2(2 - m + c)}\right.\right.$ &c.

Page 291, line 9, read $+\frac{3}{16}\frac{a}{a_i^3} e_i^2 \cos (t + 2z)$

Page 294, line 20, read $+\frac{m_i a}{2a_i^2} \cos (2nt - n_i t - \varpi) + \frac{2m_i a}{a_i^2} e_i \cos (nt - 2n_i t + \varpi_i)$

## XVII. *On a peculiar class of Acoustical Figures; and on certain Forms assumed by groups of particles upon vibrating elastic Surfaces.* By M. FARADAY, *F.R.S. M.R.I., Corr. Mem. Royal Acad. Sciences of Paris, &c. &c.*

Read May 12, 1831.

1. THE beautiful series of forms assumed by sand, filings, or other grains, when lying upon vibrating plates, discovered and developed by CHLADNI, are so striking as to be recalled to the minds of those who have seen them by the slightest reference. They indicate the quiescent parts of the plates, and visibly figure out what are called the nodal lines.

2. Afterwards M. CHLADNI observed that shavings from the hairs of the exciting violin bow did not proceed to the nodal lines, but were gathered together on those parts of the plate the most violently agitated, i. e. at the centres of oscillation. Thus when a square plate of glass held horizontally was nipped above and below at the centre, and made to vibrate by the application of a violin bow to the middle of one edge, so as to produce the lowest possible sound, sand sprinkled on the plate assumed the form of a diagonal cross; but the light shavings were gathered together at those parts towards the middle of the four portions where the vibrations were most powerful and the excursions of the plate greatest.

3. Many other substances exhibited the same appearance. Lycopodium, which was used as a light powder by OERSTED, produced the effect very well. These motions of lycopodium are entirely distinct from those of the same substance upon plates or rods in which longitudinal vibrations are excited.

4. In August 1827, M. SAVART read a paper to the Royal Academy of Sciences *, in which he deduced certain important conclusions respecting the subdivision of vibrating sonorous bodies from the forms thus assumed by light powders. The arrangement of the sand into lines in CHLADNI's experiments

* Annales de Chimie, xxxvi. p. 187.

shows a division of the sounding plate into parts, all of which vibrate isochronously, and produce the same tone. This is the principal mode of division. The fine powder which can rest at the places where the sand rests, and also accumulate at other places, traces a more complicated figure than the sand alone, but which is so connected with the first, that, as M. Savart states, "the first being given, the other may be anticipated with certainty; from which it results that every time a body emits sounds, not only is it the seat of many modes of division which are superposed, but amongst all these modes there are always two which are more distinctly established than all the rest. My object in this memoir is to put this fact beyond a doubt, and to study the laws to which they appear subject."

5. M. Savart then proceeds to establish a secondary mode of division in circular, rectangular, triangular and other plates; and in rods, rings, and membranes. This secondary mode is pointed out by the figures delineated by the lycopodium or other light powder; and as far as I can perceive, its existence is assumed, or rather proved, exclusively from these forms. Hence much of the importance which I attach to the present paper. A secondary mode of division, so subordinate to the principal as to be always superposed by it, might have great influence in reasonings upon other points in the philosophy of vibrating plates; to prove its existence therefore is an important matter. But its existence being assumed and supported by such high authority as the name of Savart, to prove its non-existence, supposing it without foundation, is of equal consequence.

6. The essential appearances, as far as I have observed them, are as follows. Let the plate before mentioned (2), which may be three or four inches square, be nipped and held in a horizontal position by a pair of pincers of the proper form, and terminated, at the part touching the glass, by two pieces of cork; let lycopodium powder be sprinkled over the plate, and a violin bow be drawn downwards against the middle of one edge so as to produce a clear full tone. Immediately the powder on those four parts of the plate towards the four edges will be agitated, whilst that towards the two diagonal cross lines will remain nearly or quite at rest. On repeating the application of the bow several times, a little of the loose powder, especially that in small masses, will collect upon the diagonal lines, and thus, showing one of the figures which Chladni dis-

covered, will also show the principal mode of division of the plate. Most of the powder which remains upon the plate will, however, be collected in four parcels; one placed near to each edge of the plate, and evidently towards the place of greatest agitation. Whilst the plate is vibrating (and consequently sounding) strongly, these parcels will each form a rather diffuse cloud, moving rapidly within itself; but as the vibration diminishes, these clouds will first contract considerably in bulk, and then settle down into four groups, each consisting of one, two, or more hemispherical parcels (53), which are in an extraordinary condition; for the powder of each parcel continues to rise up at the centre and flow down on every side to the bottom, where it enters the mass to ascend at the centre again, until the plate has nearly ceased to vibrate. If the plate be made to vibrate strongly, these parcels are immediately broken up, being thrown into the air, and form clouds, which settle down as before; but if the plate be made to vibrate in a smaller degree, by a more moderate application of the bow, the little hemispherical parcels are thrown into commotion without being sensibly separated from the plate, and often slowly travel towards the quiescent lines. When one or more of them have thus receded from the place over which the clouds are always formed, and a powerful application of the bow is made, sufficient to raise the clouds, it will be seen that these heaps rapidly diminish, the particles of which they are composed being swept away from them, and passing back in a current over the glass to the cloud under formation, which ultimately settles as before into the same four groups of heaps. These effects may be repeated any number of times, and it is evident that the four parts into which the plate may be considered as divided by the diagonal lines are repetitions of one effect.

7. The form of the little heaps, and the involved motion they acquire, are no part of the phenomena under consideration at present. They depend upon the adhesion of the particles to each other and to the plate, combined with the action of the air or surrounding medium, and will be resumed hereafter (53). The point in question is the manner in which fine particles do not merely remain at the centres of oscillation, or places of greatest agitation, but are actually driven towards them, and that with so much the more force as the vibrations are more powerful.

8. That the agitated substance should be in very fine powder, or very light, appears to be the only condition necessary for success; fine scrapings from a

common quill, even when the eighth of an inch in length or more, will show the effect. Chemically pure and finely divided silica rivals lycopodium in the beauty of its arrangement at the vibrating parts of the plate, although the same substance in sand or heavy particles proceeds to the lines of rest. Peroxide of tin, red lead, vermilion, sulphate of baryta, and other heavy powders when highly attenuated, collect also at the vibrating parts. Hence it is evident that the nature of the powder has nothing to do with its collection at the centres of agitation, provided it be dry and fine.

9. The cause of these effects appeared to me, from the first, to exist in the medium within which the vibrating plate and powder were placed, and every experiment which I have made, together with all those in M. Savart's paper, either strongly confirm, or agree with this view. When a plate is made to vibrate (2), currents (24) are established in the air lying upon the surface of the plate, which pass from the quiescent lines towards the centres or lines of vibration, that is, towards those parts of the plates where the excursions are greatest, and then proceeding outwards from the plate to a greater or smaller distance, return towards the quiescent lines. The rapidity of these currents, the distance to which they rise from the plate at the centre of oscillation, or any other part, the blending of the progressing and returning air, their power of carrying light or heavy particles, and with more or less rapidity or force, are dependent upon the intensity or force of the vibrations, the medium in which the vibrating plate is placed, the vicinity of the centre of vibration to the limit or edge of the plate, and other circumstances, which a simple experiment or two will immediately show must exert much influence on the phenomena.

10. So strong and powerful are these currents, that when the vibrations were energetic, the plate might be inclined 5°, 6°, or 8° to the horizon and yet the gathering clouds retain their places. As the vibrations diminished in force, the little heaps formed from the cloud descended the hill; but on strengthening the vibrations they melted away, the particles ascending the inclined plane on those sides proceeding upwards, and passing again to the cloud. This took place when neither sand nor filings could rest on the quiescent or nodal lines. Nothing could remain upon the plate except those particles which were so fine as to be governed by the currents, which (if they exist at all) it is evident would exist in whatever situation the plate was placed.

11. M. Savart seems to consider that the reason why the powder gathers together at the centres of oscillation is, "that the amplitude of the oscillations being very great, the middle of each of those centres (of vibration) is the only place where the plate remains nearly plane and horizontal, and where, consequently, the powder may reunite, whilst the surface being inclined to the right or left of this point, the parcels of powder cannot stop there." But the inclination thus purposely given to the plate, was very many times that which any part acquires by vibration in a horizontal position, and consequently proves that the horizontality of any part of the plate is not the cause of the powder collecting there, although it may be favourable to its remaining there when collected.

12. Guided by the idea of what ought to happen, supposing the cause now assigned were the true one, the following amongst many other experiments were made. A piece of card about an inch long and a quarter of an inch wide was fixed by a little soft cement on the face of the plate near one edge, the plate held as before at the middle, lycopodium or fine silica strewed upon it, and the bow applied at the middle of another edge; the powder immediately advanced close to the card, and the place of the cloud was much nearer to the edge than before. Fig. 1 represents the arrangement; the diagonal lines being those which sand would have formed, the line at the top *a* representing the place of the card, and the × to the right the place where the bow was applied. On applying a second piece of card as at *b*, the powder seemed indifferent to it or nearly so, and ultimately collected as in the first figure: *c* represents the place of the cloud when no card is present.

Fig. 1.

13. Pieces of card were then fixed on the glass in the three angular forms represented in fig. 2; upon vibrating the plate the fine powder always went into the angle, notwithstanding its difference of position in the three experiments, but perfectly in accordance with the idea of currents intercepted more or less by the card. When two pieces of card were fixed on the plate as in fig. 3. *a*, the powder proceeded into the angle but not to the edge of the glass, remaining about $\frac{1}{4}$th of an inch from it; but on closing up that opening, as at *b*, the powder went quite up into the corner.

Fig. 2.

Fig. 3.

14. Upon fixing two pieces of card on the plate as at *c* fig. 3, the powder between them collected in the middle very nearly as if no card had been present; but that on the outside of the cards gathered close up against them, being able to proceed so far in its way to the middle, but no further.

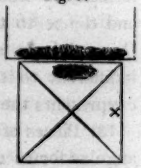
15. In all these experiments the sound was very little lowered, the form of the cross was not changed, and the light powders collected on the other three portions of the plate, exactly as if no card walls had been applied on the fourth; so that no reason appears for supposing that the mode in which the plate vibrated was altered, but the powders seem to have been carried forward by currents which could be opposed or directed at pleasure by the card stops.

16. A piece of gold-leaf being laid upon the plate, so that it did not overlap the edge, fig. 4, the current of air towards the centre of vibration was beautifully shown; for, by its force, the air crept in under the gold-leaf on all sides, and raised it up into the form of a blister; that part of the gold-leaf corresponding to the centre of the locality of the cloud, when light powder was used, being frequently a sixteenth or twelfth of an inch from the glass. Lycopodium or other fine powder sprinkled round the edge of the gold-leaf, was carried in by the entering air, and accumulated underneath.

Fig. 4.

17. When silica was placed on the edge of another glass plate, or upon a book, or block of wood, and the edge of the vibrating plate brought as nearly as possible to the edge of the former, fig. 5, part of the silica was always driven on to the vibrating plate, and collected in the usual place; as if in the midst of all the agitation of the air in the neighbourhood of the two edges, there was still a current towards the centre of vibration, even from bodies not themselves vibrating.

Fig. 5.



18. When a long glass plate is supported by bridges or strings at the two nodal lines represented in fig. 6, and made to vibrate, the lycopodium collects in three divisions; that between the nodal lines does not proceed at once into a line equidistant from the nodal lines and parallel to them, but advances from the edges of the plate towards the middle by paths, which are a little curved and oblique to the edges where they occur near the nodal lines, but are almost

Fig. 6.

perpendicular to it elsewhere, and the powder gradually forms a line along the middle of the plate; it is only by continuing the experiment for some time that it gathers up into a heap or cloud equidistant from the nodal lines. But upon fixing card walls upon this plate, as in fig. 7, the course of the powder within the cards was directly parallel to them and to the edge, instead of being perpendicular, and also directly towards the centre of oscillation. To prove that it was not as a weight that the card acted, but as an obstacle to the currents of air formed, it was not moved from its place, but bent flat down outwards, and then the fine powder resumed the courses it took upon the plate when without the cards. Upon raising the cards the first effect was reproduced.

Fig. 7.

19. The lycopodium sprinkled over the extremities of such a plate proceeds towards places equidistant from the sides and near the ends, as at *a* fig. 8; but on cementing a piece of paper to the edge, so as to form a wall about one quarter or one third of an inch high, *b*, the powder immediately moved up to it, and retained this new place. In a longer narrow plate, similarly arranged, the powder could be made to pass to either edge, or to the middle, according as paper interceptors to the currents of air were applied.

Fig. 8.

20. Plates of tin, four or five inches long, and from an inch to two inches wide, fixed firmly at one end in a horizontal position, and vibrated by applying the fingers, show the progress of the air and the light powders well. The vibrations are of comparatively enormous extent, and the appearances are consequently more instructive.

21. If a tuning-fork be vibrated, then held horizontally with the broad surface of one leg uppermost, and a little lycopodium be sprinkled upon it, the collection of the powder in a cloud along the middle, and the formation of the involving heaps also in a line along the middle of the vibrating steel bar, may be beautifully observed. But if a piece of paper be attached by wax to the side of the limb, so as to form a fence projecting above it, as in the former experiments (19), then the powder will take up its place close to the paper; and if pieces of paper be attached on different parts of the same leg, the powder will go to the different sides, in the different parts, at the same time.

22. The effects under consideration are exceedingly well shown and illus-

trated by membranes. A piece of parchment was stretched and tightly tied, whilst moist, over the aperture of a funnel five or six inches in diameter; a small hole was made in the middle, and a horse-hair passed through it, but with a knot at the extremity that it might thereby be retained. Upon fixing the funnel in an upright position, and after applying a little powdered resin to the thumbs and fore-fingers, drawing them upward over the horse-hair, the membrane was thrown into vibration with more or less force at pleasure. By supporting the funnel on a ring, passing the horse-hair in the opposite direction through the hole in the membrane, and drawing the fingers over it downwards, the direction in which the force was applied could be varied according to circumstances.

23. When lycopodium or light powders were sprinkled upon this surface, the rapidity with which they ran to the centre, the cloud formed there, the involving heaps, and many other circumstances, could be observed very advantageously.

24. The currents which I have considered as existing upon the surface of the plate, membranes, &c. from the quiescent parts towards the centres or lines of vibration (9), arise necessarily from the mechanical action of that surface upon the air. As any particular part of the surface moves upwards in the course of its vibration, it propels the air and communicates a certain degree of force to it, perpendicular or nearly so to the vibrating surface; as it returns, in the course of its vibration, it recedes from the air so projected, and the latter consequently tends to return into the partial vacuum thus formed. But as of two neighbouring portions of air, that over the part of the plate nearest to the centre of oscillation has had more projectile force communicated to it than the other, because the part of the plate urging it was moving with greater velocity, and through a greater space, so it is in a more unfavourable condition for its immediate return, and the other, i. e. the portion next to it towards the quiescent line, presses into its place. This effect is still further favoured, because the portion of air thus displaced is urged from similar causes at the same moment into the place left vacant by the air still nearer the centre of oscillation; so that each time the plate recedes from the air, an advance of the air immediately above it is made from the quiescent towards the vibrating parts of the plates.

25. It will be evident that this current is highly favourable for the transference of light powders towards the centre of vibration. Whilst the air is forced forward, the advance of the plate against the particles holds them tight; but when the plate recedes, and the current exists, the particles are at that moment left unsupported except by the air, and are free to move with it.

26. The air which is thus thrown forward at and towards the centre of oscillation, must tend by the forces concerned to return towards the quiescent lines, forming a current in the opposite direction to the first, and blending more or less with it. I endeavoured, in various ways, to make the extent of this system of currents visible. In the experiment already referred to, where gold-leaf was placed over the centre of oscillation (16), the upward current at the most powerful part was able to raise the leaf about one tenth of an inch from the plate. The higher the sounds with the same plate or membrane, i. e. the greater the number of vibrations, the less extensive must be the series of currents; the slower the vibrations, or the more extensive the excursion of the parts from increased force applied, the greater the extent of disturbance. With glass plates (2. 12) the cloud is higher and larger as the vibrations are stronger, but still not so extensive as they are upon the stretched membrane (22), where the cloud may frequently be seen rising up in the middle and flowing over towards the sides.

27. When the membrane stretched upon the funnel (22) was made to vibrate by the horse-hair proceeding downwards, and a large glass tube, as a cylindrical lamp-glass, was brought near to the centre of vibration, no evidence of a current entirely through the lamp-glass could be perceived; but still the most striking proofs were obtained of the existence of carrying currents by the effects upon the light powder, for it flew more rapidly under the edge, and tended to collect towards the axis of the tube; it could even be diverted somewhat from its course towards the centre of oscillation. A piece of upright paper, held with its edge equally near, did not produce the same effect; but immediately that it was rolled into a tube, it did. When the glass chimney was suspended very carefully, and at but a small distance from the membrane, the powder often collected at the edge, and revolved there; a complicated action between the currents and the space under the thickness of the glass taking place, but still tending to show the influence of the air in arranging and disposing the powders.

28. A sheet of drawing-paper was stretched tightly over a frame so as to form a tense elastic surface nearly three feet by two feet in extent. Upon placing this in a horizontal position, throwing a spoonful of lycopodium upon it, and striking it smartly below with the fingers, the phenomena of collection at the centre of vibration, and of moving heaps, could be obtained upon a magnificent scale. When the lycopodium was uniformly spread over the surface, and any part of the paper slightly tapped by the hand, the lycopodium at any place chosen could be drawn together merely by holding the lamp-glass over it. It will be unnecessary to enter into the detail of the various actions combining to produce these effects; it is sufficiently evident, from the mode in which they may be varied, that they depend upon currents of air.

29. A very interesting set of effects occurred when the stretched parchment upon the funnel (22) was vibrated under plates; the horse-hair was directed downwards, and the membrane, after being sprinkled over with light powder, was covered by a plate of glass resting upon the edge of the funnel; upon throwing the membrane into a vibratory state, the powder collected with much greater rapidity than without the plate; and instead of forming the semi-globular moving heaps, it formed linear arrangements, all concentric to the centre of vibration. When the vibrations were strong, these assumed a revolving motion, rolling towards the centre at the part in contact with the membrane, and from it at the part nearest the glass; thus illustrating in the clearest manner the double currents caged up between the glass and the membrane. The effect was well shown by carbonate of magnesia.

30. Sometimes when the plate was held down very close and tight, and the vibrations were few and large, the powder was all blown out at the edge; for then the whole arrangement acted as a bellows; and as the entering air travelled with much less velocity than the expelled air, and as the forces of the currents are as the squares of the velocity, the issuing air carried the powder more forcibly than the air which passed in, and finally threw it out.

31. A thin plate of mica laid loosely upon the vibrating membrane showed the rotating concentric lines exceedingly well.

32. From these experiments on plates and surfaces vibrating in air, it appears that the forms assumed by the determination of light powders towards the places of most intense vibration, depend, not upon any secondary mode of

division, or upon any immediate and peculiar action of the plate, but upon the currents of air necessarily formed over its surface, in consequence of the extra-mechanical action of one part beyond another. In this point of view the nature of the medium in which those currents were formed ought to have great influence over the phenomena; for the only reason why silica as sand should pass towards the quiescent lines, whilst the same silica as fine powder went from them, is, that in its first form the particles are thrown up so high by the vibrations as to be above the currents, and that if they were not thus thrown out of their reach they would be too heavy to be governed by them; whilst in the second form they are not thrown out of the lower current, except near the principal place of oscillation, and are so light as to be carried by it in whatever direction it may proceed.

33. In the exhausted receiver of the air-pump therefore the phenomena ought not to occur as in air; for as the force of the currents would be there excessively weakened, the light powders ought to assume the part of heavier grains in the air. Again, in denser media than air, as in water for instance, there was every reason to expect that the heavier powder, as sand and filings, would perform the part of light powders in air, and be carried from the quiescent to the vibrating parts.

34. The experiments in the air-pump receiver were made in two ways. A round plate of glass was supported on four narrow cork legs upon a table, and then a thin glass rod with a rounded end held perpendicularly upon the middle of the glass. By passing the moistened fingers longitudinally along this rod the plate was thrown into a vibratory state; the cork legs were then adjusted in the circular nodal line occurring with this mode of vibration; and when their places were thus found they were permanently fixed. The plate was then transferred into the receiver of an air-pump, and the glass rod by which it was to be thrown into vibration passed through collars in the upper part of the receiver, the entrance of air there being prevented by abundance of pomatum. When fine silica was sprinkled upon the plate, and the plate vibrated by the wet fingers applied to the rod, the receiver not being exhausted, the fine powder travelled from the nodal line, part collecting at the centre, and other part in a circle, between the nodal line and the edge. Both these situations were places of vibration, and exhibited themselves as such by the agitation of the powder. Upon again sprinkling fine silica uniformly over the plate, ex-

hausting the receiver to twenty-eight inches, and vibrating the plate, the silica went from the middle towards the nodal line or place of rest, performing exactly the part of sand in air. It did not move at the edges of the plate, and as the apparatus was inconvenient and broke during the experiment, the following arrangement was adopted in its place.

35. The mouth of a funnel was covered (22) with a well-stretched piece of fine parchment, and then fixed on a stand with the membrane horizontal; the horse-hair was passed loosely through a hole in a cork, fixed in a metallic tube on the top of the air-pump receiver; the tube above the cork was filled to the depth of half an inch with pomatum, and another perforated cork put over that; a cup was formed on the top of the second cork, which was filled with water. In this way the horse-hair passed first through pomatum and then water, and by giving a little pressure and rotatory motion to the upper cork during the time that the horse-hair was used to throw the membrane into vibration, it was easy to keep the pomatum below perfectly in contact with the hair, and even to make it exude upwards into the water above. Thus no possibility of the entrance of air by and along the horse-hair could exist, and the tightness of all the other and fixed parts of the apparatus was ascertained by the ordinary mode of examination. A little paper shelf was placed in the receiver under the cork to catch any portion of pomatum that might be forced through by the pressure, and prevent its falling on to the membrane.

36. This arrangement succeeded: when the receiver was full of air, the lycopodium gathered at the centre of the membrane with great facility and readiness, exhibiting the cloud, the currents, and the involving heaps. Upon exhausting the receiver until the barometrical gauge was at twenty-eight inches, the lycopodium, instead of collecting at the centre, passed across the membrane towards one side which was a little lower than the other. It passed by the middle just as it did over any other part; and when the force of the vibrations was much increased, although the powder was more agitated at the middle than elsewhere, it did not collect there, but went towards the edges or quiescent parts. Upon allowing air to enter until the barometer stood at twenty-six inches, and repeating the experiments, the effect was nearly the same. When the vibrations were very strong, there were faint appearances of a cloud, consisting of the very finest particles, collecting at the centre of vibration;

but no sensible accumulation of the powder took place. At twenty-four inches of the barometer the accumulation at the centre began to appear, and there was a sensible, though very slight effect visible of the return of the powder from the edges. At twenty-two inches these effects were stronger; and when the barometer was at twenty inches, the currents of air within the receiver had force enough to cause the collection of the principal part of the lycopodium at the centre of vibration. Upon again, however, restoring the exhaustion to twenty-eight inches, all the effects were reproduced as at first, and the lycopodium again proceeded to the lower or the quiescent parts of the membrane. These alternate effects were obtained several times in succession before the apparatus was dismounted.

37. In this form of experiment there were striking proofs of the existence of a current upwards from the middle of the membrane when vibrating in air, (24), and the extent of the system of currents (26) was partly indicated. The powder purposely collected at the middle by vibrations, when the receiver was full of air, was observed as to the height to which it was forced upwards by the vibrations; and then the receiver being exhausted, the height to which the powder was thrown by similar vibrations was again observed. In the latter cases it was nothing like so great as in the former, the height not being two-thirds, and barely one-half, the first height. Had the powder been thrown up by mere propulsion, it should have risen far higher in vacuo than in air: but the reverse took place; and the cause appears to be, that in air the current had force enough to carry the fine particles up to a height far beyond what the mere blow which they received from the vibrating membrane could effect.

38. For the experiments in a denser medium than air, water was chosen. A circular plate of glass was supported upon four feet in a horizontal position, surrounded by two or three inches of water, and thrown into vibration by applying a glass rod perpendicular to the middle, as in the first experiment in vacuo (34); the feet were shifted until the arrangement gave a clear sound, and the moistened brass filings sprinkled upon the plate formed regular lines or figures. These lines were not however lines of rest, as they would have been in the air, but were the places of greatest vibration; as was abundantly evident from their being distant from that nodal line determined and indicated by the contact of the feet, and also from the violent agitation of the filings.

In fact, the filings proceeded from the quiescent to the moving parts, and there were gathered together; not only forming the cloud of particles over the places of intense vibration, but also settling down, when the vibrations were weaker, into the same involving groups, and in every respect imitating the action of light powders in air. Sand was affected exactly in the same manner; and even grains of platina could be in this way collected by the currents formed in so dense a medium as water.

39. The experiments were then made under water with the membranes stretched over funnels (22) and thrown into vibration by horse-hairs drawn between the fingers. The space beneath the membrane could be retained, filled with air, whilst the upper surface was covered two or three inches deep with water; or the space below could also be filled with water, or the force applied to the membrane by the horse-hair could be upwards or downwards at pleasure. In all these experiments the sand or filings could be made to pass with the utmost facility to the most powerfully vibrating part, that being either at the centre only, or in addition, in circular lines, according to the mode in which the membrane vibrated. The edge of the funnel was always a line of rest; but circular nodal lines were also formed, which were indicated, not by the accumulation of filings upon them, but by the tranquil state of those filings which happened to be there, and also by being between those parts where the filings, by their accumulation and violent agitation, indicated the parts in the most powerful vibratory state.

40. Even when by the relaxation of the parchment from moisture, and the force upwards applied by the horse-hair, the central part of the membrane was raised the eighth of an inch or more above the edges, the circle not being four inches in diameter, still the filings would collect there.

41. When in place of parchment common linen was used, as becoming tighter rather than looser when wetted, the same effects were obtained.

42. Both the reasoning adopted and the effects described were such as to lead to the expectation that if the plate vibrating in air was covered with a layer of liquid instead of sand or lycopodium, that liquid ought to be determined from the quiescent to the vibrating parts and be accumulated there. A square plate was therefore covered with water, and vibrated as in the former experiments (2. 6.); but all endeavours to ascertain whether accumulation

occurred at the centres of oscillation, either by direct observation, or the reflection from its surface of right-lined figures, or by looking through the parts, as through a lens, at small print and other objects, failed.

43. As however when the plate was strongly vibrated, the well-known and peculiar crispations which form on water at the centres of vibration, occurred and prevented any possible decision as to accumulation, it was only when these were absent and the vibration weak, and the accumulation therefore small, that any satisfactory result could be expected; but as even then no appearance was perceived, it was concluded that the force of gravity combined with the mobility of the fluid was sufficient to restore the uniform condition of the layer of water after the bow was withdrawn, and before the eye had time to observe the convexity expected.

44. To remove in part the effect of gravity, or rather to make it coincide with, instead of oppose the convexity, the under surface of the plate was moistened instead of the upper, and by inclining the plate a little, the water made to hang in drops at *a* or *b* or *c*, fig. 9, at pleasure. On applying the bow at ×, and causing the plate to vibrate, the drops instantly disappeared, the water being gathered up and expanded laterally over the parts of the plate from which it had flowed. On stopping the vibration, it again accumulated in hanging drops, which instantly disappeared as before on causing the plate to vibrate, the force of gravity being entirely overpowered by the superior forces excited by the vibrating plate. Still, no visible evidence of convexity at the centres of vibration were obtained, and the water appeared rather to be urged from the vibrating parts than to them.

Fig. 9.

45. The tenacity of oil led to the expectation that better results would be obtained with it than with water. A round plate, held horizontally by the middle (6. 42), was covered with oil over the upper surface, so as to be flooded, except at ×, fig. 10, and the bow applied at × as before, to produce strong vibration. No crispation occurred in the oil, but it immediately accumulated at *a*, *b*, and *c*, forming fluid lenses there, rendered evident by their magnifying power when print was looked at through them. The accumulations were also visible on putting a sheet of white paper beneath, in consequence of the colour of the oil

Fig. 10.

being deeper at the accumulations than elsewhere; and they were also rendered beautifully evident by making the experiment in sunshine, or by putting a candle beneath the plate, and placing a screen on the opposite side to receive the images formed at the focal distance.

46. When the vibration of the plate ceased, the oil gradually flowed back until of uniform depth. On renewing the vibration, the accumulations were re-formed, the phenomena of accumulation occurring with as much certainty and beauty as if lycopodium powder had been used.

47. To remove every doubt of the fluid passing from the quiescent to the agitated parts, centres of vibration were used, nearly surrounded by nodal lines. A square plate, fig. 11, being held at *c*, and the bow applied at ×, gave with sand, nodal lines, resembling those in the figure. Then clearing off the sand, putting oil in its place, and producing the same mode of vibration as before, the oil accumulated at *a* and *b*, forming two heaps or lenses as in the former experiment (45).

Fig. 11.

48. The experiment made with water on the under surface (44) was now repeated with oil, the round plate being used (45). The hanging drop of oil rose up as the water did before, but the lateral diffusion was soon limited; for lenses were formed at the centres of vibration just as when the oil was upon the upper surface, and, as far as could be ascertained by general examination, of the same form and power. On stopping the vibration, the oil gathered again into hanging drops; and on renewing it, it was again disposed in the lens-like accumulations.

49. With white of egg the same observable accumulation at the centres of vibration could be produced.

50. Hence it is evident that when a surface vibrating normally, is covered with a layer of liquid, that liquid is determined from the quiescent to the vibrating parts, producing accumulation at the latter places; and that this accumulation is limited, so that if purposely rendered too great by gravity or other means, it will quickly be diminished by the vibrations until the depth of fluid at any one part has a certain and constant relation to the velocity there and to the depth elsewhere.

51. From the accumulated evidence which these experiments afford, I think there can remain no doubt of the cause of the collection of fine powders at the

centres or lines of vibration of plates, membranes, &c. under common circumstances; and that no secondary mode of division need be assumed to account for them. I have been the more desirous of accumulating experimental evidence, because I have thought on the one hand that the authority of Savart should not be doubted on slight grounds, and on the other, that if by accident it be placed in the wrong scale, the weight of evidence against it should be such as fully to establish the truth and prevent a repetition of the error by others.

52. It must be evident that the phenomena of collection at the centres or lines of greatest vibration are exhibited in their purest form at those places which are surrounded by nodal lines; and that where the centre or place of vibration is at or near to an edge, the effects must be very much modified by the manner in which the air is there agitated. It is this influence, which, in the square plates (6. 12) and other arrangements, prevents the clouds being at the very edge of the glass. They may be well illustrated by vibrating tin plates under water over a white bottom, and sprinkling dark-coloured sand or filings upon various parts of the plates.

*On the peculiar Arrangement and Motions of the heaps formed by particles lying on vibrating surfaces.*

53. The peculiar manner in which the fine powder upon a vibrating surface is accumulated into little heaps, either hemispherical or merely rounded, and larger or smaller in size, has already been described (6. 28), as well also as the singular motion which they possess, as long as the plate continues in vibration. These heaps form on any part of the surface which is in a vibratory state, and not merely under the clouds produced at the centres of vibration, although the particles of the clouds always settle into similar heaps. They have a tendency, as heaps, to proceed to the nodal or quiescent lines, but are often swept away in powder by the currents already described (6). When on a place of rest, they do not acquire the involving motion. When two or more are near together or touch, they will frequently coalesce and form but one heap, which quickly acquires a rounded outline. When in their most perfect and final form, they are always round.

54. The moving heaps formed by lycopodium on large stretched drawing-

paper (28), are on so large a scale as to be very proper for critical examination. The phenomena can be exhibited also even by dry sand on such a membrane, the sand being in large quantity and the vibrations slow. When the surface is thickly covered by sand from a sieve, and the paper tapped with the finger, the manner in which the sand draws up into moving heaps is very beautiful.

55. When a single heap is examined, which is conveniently done by holding a vibrating tuning-fork in a horizontal position, and dropping some lycopodium upon it, it will be seen that the particles of the heap rise up at the centre, overflow, fall down upon all sides, and disappear at the bottom, apparently proceeding inwards; and this evolving and involving motion continues until the vibrations have become very weak.

56. That the medium in which the experiment is made has an important influence, is shown by the circumstance of heavy particles, such as filings, exhibiting all these peculiarities when they are placed upon surfaces vibrating in water (39); the heaps being even higher at the centre than a heap of equal diameter formed of light powder in the air. In water, too, they are formed indifferently upon any part of the plate or membrane which is in a vibratory state. They do not tend to the quiescent lines; but that is merely from the great force of the currents formed in water as already described (38), and the power with which they urge obstacles to the place of greatest vibration.

57. If a glass plate be supported and vibrated (6), its surface having been covered with sand enough to hide the plate, and water enough to moisten and flow over the sand, the sand will draw together in heaps, and these will exhibit the peculiar and characteristic motion of the particles in a very striking manner.

58. The aggregation and motion of these heaps, either in air or other fluids, is a very simple consequence of the mechanical impulse communicated to them by the joint action of the vibrating surface and the surrounding medium. Thus in air, when, in the course of a vibration, the part of a plate under a heap rises, it communicates a propelling force upwards to that heap, mingled as it is with air, greater than that communicated to the surrounding atmosphere, because of the superior specific gravity of the former; upon receding from the heap, therefore, in performing the other half of its vibration, it forms a partial

vacuum, into which the air, round the heap, enters with more readiness than the heap itself; and as it enters, carries in the powder at the bottom edge of the heap with it. This action is repeated at every vibration, and as they occur in such rapid succession that the eye cannot distinguish them, the centre part of the heap is continually progressing upwards; and as the powder thus accumulates above, whilst the base is continually lessened by what is swept in underneath, the particles necessarily fall over and roll down on every side.

59. Although this statement is made upon the relation of the heap, as a mass, to the air surrounding it, yet it will be seen at once that the same relation exists between any two parts of the heap at different distances from the centre; for the one nearest the centre will be propelled upward with the greatest force, and the other will be in the most favourable state for occupying the partial vacuum left by the receding plate.

60. This view of the effect will immediately account for all the appearances; the circular form, the fusion together of two or more heaps, their involving motion, and their existence upon any vibrating part of the plate. The manner in which the neighbouring particles would be absorbed by the heaps is also evident; and as to their first formation, the slightest irregularities in the powder or surface would determine a commencement, which would then instantly favour the increase.

61. It is quite true, that if the powder were coherent, that force alone would tend to produce the same effect, but only in a very feeble degree. This is sufficiently shown by the experiments made in the exhausted receiver (36). When the barometer of the air-pump was at twenty-eight inches, that in the air being about 29.2 inches, the heaps, or rather parcels, formed very beautifully over the whole surface of the membrane; but they were very flat and extensive compared with the heaps in air, and the involving motion was very weak. As the air was admitted, the vibration being continued, the heaps rose in height, contracted in diameter, and moved more rapidly. Again, in the experiments with filings and sand in water, no cohesive action could assist in producing the effect; it must have been entirely due to the manner in which the particles were mechanically urged in a medium of less density than themselves.

62. The conversion of these round heaps into linear concentric involving parcels, in the experiment already described (29. 31), when the membrane was

covered by a plate of glass, is a necessary consequence of the arrangements there made, and tends to show how influential the action of the air or other including medium is in all the phenomena considered in this paper. No incompatible principles are assumed in the explication given of the arrangement of the forces producing the two classes of effects in question, and though by variation of the force of vibration and other circumstances, the one effect can be made, within certain limits, to pass into the other, no anomaly or contradiction is thus involved, nor any result produced, which, as it appears to me, cannot be immediately accounted for by reference to the principles laid down.

*Royal Institution,*
*March* 21, 1831.

## APPENDIX.

### *On the Forms and States assumed by Fluids in contact with vibrating elastic surfaces.*

63. When the upper surface of a plate vibrating so as to produce sound (2. 6) is covered with a layer of water, the water usually presents a beautifully crispated appearance in the neighbourhood of the centres of vibration. This appearance has been observed by Oersted*, Wheatstone †, Weber‡, and probably others. It, like the former phenomena which I have endeavoured to explain, has led to false theory, and being either not understood or misunderstood, has proved an obstacle to the progress of acoustical philosophy.

64. On completing the preceding investigation, I was led to believe that the principles assumed would, in conjunction with the cohesion of fluids, account for these phenomena. Experimental investigation fully confirmed this expectation, but the results were obtained at too late a period to be presented to the Royal Society before the close of the Session; and it is only because the philosophy and the subject itself is a part of that received into the Philosophical Transactions in the preceding paper, that I am allowed, by the President and Council, the privilege of attaching the present paper in the form of an Appendix.

65. The general phenomenon now to be considered is easily produced upon a square plate nipped in the middle, either by the fingers or the pincers (2. 6), held horizontally, covered with sufficient water on the upper surface to flow freely from side to side when inclined, and made to vibrate strongly by a bow applied to one edge, ×, fig. 12, in the usual way. Crispations appear on the surface of the water, first at the centres of vibration, and extend more or less towards the nodal lines, as the vibrations are stronger or weaker. The crispation presents the appearance of small conoidal elevations of equal lateral extent, usually arranged

Fig. 12.

* Lehrbe's Hist. of Natural Phenomena for 1813. † Annals of Philosophy, N. S. vi. p. 82.
‡ Wellenlehre, p. 414.

rectangularly with extreme regularity; permanent* (in appearance), so long as a certain degree of vibration is sustained; increasing and diminishing in height, with increased or diminished vibration; but not affected in their lateral extent by such variations, though the whole crispated surface is enlarged or diminished at those times. If the plate be vibrated, so as to produce a different note, the crispations still appear at the centre of vibration, but are smaller for a high note, larger for a low one. The same note produced on different sized plates, by different modes of vibration, appears to produce crispations of the same dimension, other circumstances being the same.

66. These appearances are beautifully seen when ink diluted with its bulk of water is used on the plate.

67. It was necessary, for examination, both to prolong and enlarge the effect, and the following were found advantageous modes of producing it. Plates of crown-glass, from eighteen to twenty-two inches long, and three or four inches wide, were supported each by two triangular pieces of wood acting as bridges (18), and made to vibrate by a small glass rod or tube resting perpendicularly at the middle, over which the moist fingers were passed. By sprinkling dry sand on the plates, and shifting the bridges, the nodal lines were found (usually about one fifth of the whole length from each end), and their places marked by a file or diamond. Then clearing away the sand, putting water or ink upon the plate, and applying the rod or fingers, it was easy to produce the crispations and sustain them undisturbed, and with equal intensity for any length of time.

68. By making a broad mark, or raising a little ledge of bee's wax, or a mixture of bee's wax and turpentine, it was easy to confine the pool of water to the middle part of the plate, fig. 13, where, of course, the crispations were most powerfully produced. Such a barrier is often useful to separate the wet and dry parts of the glass, especially when a violin bow is used as the exciter.

Fig. 13.

69. In other experiments, deal laths, two, three, or four feet long, one inch and a half wide, and three eighths or more of an inch in thickness, were used instead of the glass plates. These could be made to vibrate by the fingers and wet rod (67), and by either shifting the bridges or changing the lath an almost

* Weber's Wellenlehre, p. 414.

unlimited change of isochronous vibrations, from that producing a high note to those in which not more than five or six occurred in a second, could be obtained. The crispations were formed upon a glass plate attached to the middle of the lath, by two or three little pellets of soft cement*.

70. Obtained in this way the appearances were very beautiful, and the facilities very great. A glass plate, from four to eight inches square, could be covered uniformly with crispations of the utmost regularity; for, by attaching the plate with a little method, and at points equidistant from the centre of the bar, it was easy to make every part travel with the same velocity, and in that respect differ from and surpass the bar which sustained it. The conoidal heaps constituting the crispation could be so enlarged by slowness of vibration, that three or four occupied a linear inch. The glass plate could be removed, and another of different form or substance, and with other fluids, as mercury, &c., substituted in an instant.

71. In using laths, it is necessary to confine the parts bearing upon the bridges, either by slight pressure of the fingers, or by loops of string, or by weights. The exciting glass rod need not necessarily rest upon the middle of the bar or plate, but may be applied with equal effect at some distance from it. Long laths may be made to subdivide in their mode of vibration, according as the rod is applied to different places, and the pressure given by the exciting moist fingers is varied; with each change of this kind an immediate change of the crispation is observed.

72. This form of apparatus was enlarged until a board eighteen feet long was used, the layer of water being now three fourths of an inch in depth and twenty-eight inches by twenty inches in extent. The sides of the cistern were very much inclined, so that the water should gradually diminish in depth, and thus reflected waves be prevented. The vibrations were so slow as to be produced by the direct application of the hand, and the heaps were each from an inch to two inches in extent. Though of this magnitude, they were identical in their nature with those forming crispations on so small a scale as to appear merely like a dullness on the surface of the water.

73. In these experiments the proportion of water requires a general adjustment, the crispations being produced more readily and beautifully when there

* Equal parts of yellow wax and turpentine.

is a certain quantity than when there is less. For small crispations, the water should flow upon the surface freely. Large crispations require more water than small ones. Too much water sometimes interferes with the beauty of the appearance, but the crispation is not incompatible with much fluid, for the depth may amount to eight, ten, or twelve inches (111), and is probably unlimited.

74. These crispations are equally produced upon the under with the upper surface of vibrating plates. When the lower surface is moistened, and the bow applied (65), the drops which hang down by the force of gravity are rippled; but being immediately gathered up as described in the former paper (44), a certain definite layer is produced, which is beautifully rippled or crispated at the centres of vibration.

75. Most fluids, if not all, may be used to produce these crispations, but some with particular advantages; alcohol, oil of turpentine, white of egg*, ink, and milk produce them. White of egg, notwithstanding its viscosity, shows them readily and beautifully. Ink has great advantages, because, from its colour and opacity, the surface form is seen undisturbed by any reflection from the glass beneath; its appearance in sunshine is exceedingly beautiful. When diluted ink is used for large crispations, upon tin plate or over white paper, or mercury, the different degrees of colour or translucency corresponding to different depths of the fluid, give important information relative to the true nature of the phenomena (78. 85. 97). Milk is, for its opacity, of similar advantage, especially when a light is placed beneath, and being more viscid than water is better for large arrangements (72. 98), because it produces less splashing.

76. Oil does not show small crispations readily (120), and was supposed to be incapable of forming them, but when warmed (by which its liquidity is increased) it produces them freely. Cold oil will also produce large crispations, and for very large ones would probably be better than water, because of its cohesion. The difference between oil and white of egg is remarkable; for the latter, from common observation, would appear to be a thicker fluid than oil: but the qualities of cohesion differ in the two, the apparent thickness of white of egg depending upon an elastic power (probably due to an approach to

* Wheatstone.

structure), which tends to restore its particles to their first position, and co-existing with great freedom to move through small spaces, whilst that of oil is due to a real difficulty in removing the particles one by another. It is possible that the power of assuming, more or less readily, the crispated state may be a useful and even important indication of the internal constitution of different fluids.

77. With mercury the crispations are formed with great facility, and of extreme beauty, when a piece of amalgamated tin or copper plate being fixed on a lath (69), is flooded with the fluid metal, and then vibrated. A film quickly covers the metal, and then the appearances are not so regular as at first; but on removing the film by a piece of paper, their regularity and beauty are restored. It is more convenient to cover the mercury with a little very dilute acetic or nitric acid; for then the crispations may be produced and maintained for any length of time with a surface of perfect brilliancy.

78. When a layer of ink was put over the mercury, the acid of the ink removed all film, and the summits of the metallic heaps, by diminishing the thickness of the ink over them, became more or less visible, producing the appearance of pearls of equal size beautifully arranged in a black medium. When mercury covered with a film of dilute acid was vibrated in the sunshine, and the light reflected from its surface received on a screen, it formed a very beautiful and regular image; but the screen required to be placed very near to the metal, because of the short focal lengths of the depressions on the mercurial surface.

79. It is sometimes difficult to arrive by inspection at a satisfactory conclusion of the forms and arrangements thus presented, because of multiplied reflection and the particular condition of the whole, which will be described hereafter (95). When observed, well formed with vibrations so slow as to produce three or four elevations in a linear inch (70), they are seen to be conoidal heaps rounded above, and apparently passing into each other below by a curvature in the opposite direction. When arranged regularly, each is surrounded by eight others, so that, a single light being used, nine images may be sent from each elevation to the eye. These are still further complicated, when transparent fluids are used, by reflections from the glass beneath. The use of ink

(75) removes a good deal of the difficulty experienced, and the production of slow, regular, sustained vibrations, more (67. 69).

80. These elevations I will endeavour to distinguish henceforth by the term *heaps*.

81. The crispation on the long plate of glass described (67) always ultimately assumed a rectangular arrangement, i. e. the heaps were equidistant, and in rows parallel or at right angles to each other. The rows usually form angles of 45° to the sides of the plate at the commencement; but if the vibration be continued, the whole system usually wheels round through 45° until the rows coincide with the edges of the plate.

82. The lateral dimension of the heaps remained constant notwithstanding considerable variations in the force of vibration. But it was soon found that variation in the depth of water affected their number; that with less water the heaps were smaller, and with more water larger, though the sound and therefore the number of vibrations in a given period remained the same. The number of heaps could be reduced to eight or increased to eleven and a half in the three inches by a change in no other condition than the depth of fluid.

83. With the above plate (67. 81) the appearances were usually in the following order, the pool of water being quadrangular or nearly so, and the exciting rod resting in the middle of it. Ring-like linear heaps concentric to the exciting rod first form to the number of six or seven; these may be retained by a moderated state of vibration, and produce intervals which measured across the diameter of the rings are to the number of ten in three inches, with a certain constant depth of water. By increasing the force of vibration the altitude of these elevations increases, but not their lateral dimension, and then linear heaps form across these circles and the plate, and parallel to the bridges, having an evident relation to the manner in which the whole plate vibrates. These, which like all other of these phenomena are strongest at the part most strongly vibrating, soon break up the circles, and are themselves broken up, producing independent heaps, which at first are irregular and changeable, but soon become uniform and produce the quadrangular order; first at angles of 45° to the edges of the plate, but gradually moving round until parallel to them. So the arrangement continues, unless the force be so violent as to break

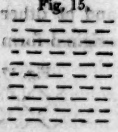
it up altogether: if the vibratory force be gradually diminished, then the heaps as gradually fall, but without returning through the order in which they were produced. The following lines may serve to indicate the course of the phenomena.

Fig. 14.

When perfectly formed, the heaps are also to the number of ten in three inches with the same depth of water as that which produced the rings. The intervals between the rings and the heaps are the same, other influential circumstances remaining unaltered.

84. Then another form of heaps occasionally occurred, but always passing ultimately into those described. These heaps were grouped in an arrangement still very nearly rectangular, and at angles of 45° to the sides of the plate, but were contracted in one direction, and elongated in the other; these directions being parallel to the sides and ends of the plate. If the marks in fig. 15 be supposed to represent the tops of the heaps, an idea of the whole will be obtained. Three inches along these heaps included eight, but across them it included fifteen nearly. These numbers are therefore the relation of length to breadth. But along the lines of the quadrilateral arrangement three inches included eleven heaps, which, notwithstanding the difference in form, is the same number that was produced by the same plate, with the same depths of water, when the heaps were round; therefore an equal number of heaps existed in the same area in both cases; and the departure from perfect rectangular arrangement, and also the ratio of 1 : 2, is probably due to some slight influence of the sides of the plate.

Fig. 15.



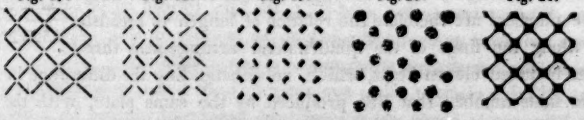
85. When mercury covered with a film of very dilute nitric acid is vibrated (77), the rectangular arrangement is constantly obtained. When vibrated under dilute ink (78), it is still more beautifully seen and distinguished. The tin plate sustaining the mercury was square, and when the whole surface was covered with crispations, the lines of the rectangular arrangement were always at angles of 45° to its edges.

86. When sand is sprinkled uniformly over a plate on which large water crispations are produced, i. e. four, five or six in the inch, it gives some very important indications. It immediately becomes arranged under the water, and with a little method may be made to yield very regular forms. It is always removed from under the heaps, passing to the parts between them, and frequently producing therefore the accompanying form, fig. 16, of great regularity. As the sand figure remains when the vibration has ceased, it allows of the determination of position, the measurement of intervals, &c. very conveniently.

Fig. 16.

87. Very often the lines of sand are not continuous, but separated with extreme regularity into portions as represented fig. 17. The portions of these lines were sometimes, with little sand on the plate, very small, fig. 18; and when more sand was present they were thickened occasionally, fig. 19; then assuming the appearance of heaps arranged in straight lines at angles of 45° to the lines regulating the position of the water-heaps which formed them, and just double in number to the latter. At other times the sand instead of being deficient at the intersecting angle would accumulate there only, fig. 20; and at other times would accumulate there principally, but still show the original form by a few connecting particles, fig. 21.

Fig. 17. Fig. 18. Fig. 19. Fig. 20. Fig. 21.



88. When the heaps were of the form described (84), the sand was still washed from under them; it did not however assume lines parallel to the rectangular arrangement of the heaps, but was arranged as in fig. 22.

Fig. 22.

89. When only the circular linear heaps (83) were produced, the sand assumed similar circular forms, concentric and alternating with the water elevations.

90. On strewing a little lycopodium over the water for the purpose of gaining information relative to what occurred at the surface during the cris-

pation, it moved about over the fluid in every possible direction, whilst the crispations existed of the utmost steadiness beneath. The same thing occurred with pieces of cork on very large crispations (98). But when much lycopodium was put on, so that the particles retained each other in a steady position, then it formed lines * parallel to the arrangement of the heaps, the powder being displaced from the parts over the heaps, and taking up an arrangement perpendicularly over the sand beneath. As the lycopodium forms float on the water they are easily disturbed, and in no respect approach as to beauty and utility to the forms produced by the sand; but lycopodium may be used with smaller crispations than sand.

91. The crispations are much influenced by various circumstances. They tend to commence at the place of greatest vibration; but if the quantity of fluid is too little there, and more abundant elsewhere, they will often commence at the latter place first. Their final arrangement is also much affected by the form of the plate, or of the pool of water on which they occur. When the plates or pools are rectangular, and all parts vibrate with equal velocity, the lines of heaps are at angles of 45° to the edges. But when semicircular and other plates were used, the arrangement, though quadrangular, was unsteady, often breaking up and starting by pieces into different and changing positions.

92. When mercury was used (77), the film formed on it after a few moments had great power, according to the manner in which it was puckered, of modifying the general arrangement of new crispations.

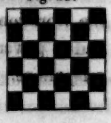
93. When a circular plate, supported by cork feet attached where a single nodal line would occur, was covered with water and vibrated by a rod resting upon the middle, the crispations extended from the middle towards the nodal line; these were sometimes arranged rectangularly, but had no steadiness of position, and changed continually. At other times the heaps appeared as if hexagonal, and were arranged hexagonally, but these also shifted continually. This and many other experiments (83) showed that the direction and nature of the vibration of the plate (i. e. of the lines of equal or varying vibrating force), had a powerful influence over the regularity and final arrangement of the crispations.

* Wheatstone.

94. The beautiful appearance exhibited when the crispations are produced in sunshine, or examined by a strong concentrated artificial light, has been already referred to (78. 79). When the reflected image from any one heap is examined, (for which purpose ink (75) or mercury (77) is very convenient,) it will be found not to be stationary, as would happen if the heap was permanent and at rest, nor yet to form a vertical line, as would occur if the heap were permanent but travelled to and fro with the vibrating plate; but it moves so as to re-enter upon its course, forming an endless figure, like those produced by Dr. Young's piano-forte wires, or Wheatstone's kaleidophone, varying with the position of the light and the observer, but constant for any particular position and velocity of vibration. Upon placing the light and the eye in positions nearly perpendicular to the general surface of the fluid, so as to avoid the direct influence of the motion of vibration, still the luminous, linear, endless figure was produced, extending more or less in different directions, according to the relation of the light and eye to the crispated surface, and occasionally corresponding in its extent one way to the width of the heap, i. e. to the distance between the summit of one heap and its neighbours, but never exceeding it. The figure produced by one heap was accurately repeated by all the heaps when the vibrating force of the plate was equal (70) and the arrangement regular.

95. The view which I had been led to anticipate of the nature of the heaps, from the effects described in the former paper, were, that each heap was a permanent elevation, like the cones of lycopodium powder (53. 58), the fluid rising at the centre, but descending down the inclined sides, the whole system being influenced, regulated, and connected by the cohesive force of the fluid. But these characters of the reflected image, with others of the effects already described, led to the conclusion, that notwithstanding the apparent permanency of the crispated surface, especially when produced on a small scale, as by the usual method, the heaps were not constant, but were raised and destroyed with each vibration of the plate; and also that the heaps did not all exist at once, but (referring to locality) formed two sets of equal number and arrangement, fig. 23, never existing together, but alternating with, and being resolved into each other, and by their rapidity of recurrence giving the appearance of simultaneous and

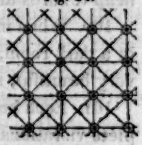
Fig. 23.



even permanent existence. Provided this view were confirmed, it seemed as if it would be easy to explain the production of the heaps, their regular arrangement, &c., and to deduce their recurrence, dimensions, and many other points relative to their condition.

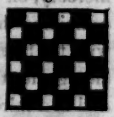
96. On producing a water crispation, having four or five heaps in a linear inch, placing a candle beneath, and a screen of French tracing paper above it, the phenomena were very beautiful, and such as supported the view taken. By placing the screen at different distances, it could be adapted to the focal length due to the curvature at different parts of the surface of fluid, so that by observing the luminous figure produced and its transitions as the screen was moved nearer or further, the general form of the surface could be deduced. Each heap with a certain distance of screen gave a star of light ⊕, fig. 24, which twinkled, i. e. appeared and disappeared alternately, as the heap rose and fell. At the corners × equidistant from these, fainter starred lights appeared; and by putting the screen nearer to or further from the surface, lines of light, in two or even four directions, appeared intersecting the luminous centres and apparently permanent, whilst circumstances remained unchanged. These effects could be magnified to almost any scale (72).

Fig. 24.



97. When heaps of similar magnitude were produced, with diluted ink on glass (75), and white paper or an illuminated screen looked at through them, a chequered appearance was observed. In one position, lines of a certain intensity separated the heaps from each other, but the square places representing the heaps looked generally lighter. In another position, when but little reflected light came from the surface of the heaps, their places could be perceived as dark, from the greater depth of ink there. By care, another position could be found in which the whole surface looked like an alternate arrangement of light and dark chequers, fig. 25, not steady, but with a quivering motion, which further attention could trace as due to a rapid alternation in which the light spaces became dark and the dark light, simultaneously. When, instead of glass, a bright tin plate was used under the diluted ink, the chequered spaces and their alternations could be seen still more beautifully.

Fig. 25.



98. It was in consequence of these effects that very large arrangements were

made (72), giving heaps that were two inches and a half wide each*; and now it was evident, by ordinary inspection, that the heaps were not stationary, but rose and fell; and also that there were two sets regularly and alternately arranged, the one set rising as the other descended.

99. Sand gave no indications of arrangement with these large heaps (86); but when some coarse saw-dust was soaked, so as to sink in water, and then distributed in the fluid, its motions were beautifully illustrative of the whole philosophy of the phenomena. It was immediately washed away from under the rising and falling heaps, and collected in the places equidistant between these spots, as the sand did in the former experiments (86), and by its vibratory motion to and fro, it showed distinctly how the water oscillated from one heap towards another, as the heaps sunk and rose.

100. When milk (75) was used instead of water for these large arrangements in a dark room, and a candle was placed beneath, the appearances also were very beautiful, resembling in character those described (97).

101. Each heap (identified by its locality) recurs or is re-formed in two complete vibrations of the sustaining surface †; but as there are two sets of heaps, a set occurs for each vibration. The maximum and minimum of height for the heaps appears to be alternately, almost immediately after the supporting plate has begun to descend in one complete vibration.

102. Many of these results are beautifully confirmed by the appearances produced, when regular crispations have been sustained for a short time with mercury, on which a certain degree of film has been allowed to form (77). On examining the film afterwards in one light, lines could be seen on it, coinciding with the intervals of the heaps in one direction; in another light, lines coinciding with the other direction came into sight, whilst the first disappeared; and in a third light, both sets of lines could be seen cutting out the square places where the heaps had existed: in these spaces the film was minutely wrinkled and bagged, as if it had there been distended; at the lines it was only a little wrinkled, giving the appearance of texture; and at the crossing

* This estimate is given in accordance with the mode of estimating the former and smaller heaps, as if the heaps were formed simultaneously; but it is evident that if only half the number exist at once, each heap will have twice the width or four times the area of those which can be formed if all exist together.

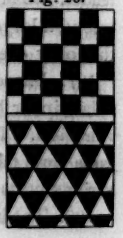
† A vibration is here considered as the motion of the plate, from the time that it leaves its extreme position until it returns to it, and not the time of its return to the intermediate position.

of the lines themselves, it was quite free from mark, and fully distended. All these are natural consequences, if the film be considered as a flexible but inelastic envelope formed over the whole surface whilst the heaps were rising and falling.

103. The mode of action by which these heaps are formed is now very evident, and is analogous in some points to that by which the currents and the involving heaps already described are produced. The plate in rising tends to lift the overlying fluid, and in falling to recede from it; and the force which it is competent to communicate to the fluid can, in consequence of the physical qualities of the latter, be transferred from particle to particle in any direction. The heaps are at their maximum elevation just after the plate begins to recede from them; before it has completed its motion downwards, the pressure of the atmosphere and that part of the force of the plate which through cohesion is communicated to them, has acted, and by the time the plate has begun to return, it meets them endowed with momentum in the opposite direction, in consequence of which they do not rise as a heap, but expand laterally, all the forces in action combining to raise a similar set of heaps, at exactly intermediate distances, which attain their maximum height just after the plate again begins to recede; these therefore undergo a similar process of demolition, being resolved into exact duplicates of the first heaps. Thus the two sets oscillate with each vibration of the plate, and the action is sustained so long as the plate moves with a certain degree of force; much of that force being occupied in sustaining this oscillation of the fluid against the resistance offered by the cohesion of the fluid, the air, the friction on the plate, and other causes.

104. A natural reason now appears for the quadrangular and right-angled arrangement which is assumed, when the crispation is most perfect. The hexagon, the square, and the equilateral triangle are the only regular figures that can fill an area perfectly. The square and triangle are the only figures that can allow of one half alternating symmetrically with the other, in conformity with what takes place between the two reciprocating sets of heaps, fig. 26; and of these two the boundary lines between squares are of shorter extent than those between equilateral triangles of equal area. It is evident therefore that one of these two will be finally assumed, and that that will be the square arrangement; because then the fluid

Fig. 26.



will offer the least resistance in its undulations to the motions of the plate, or will pass most readily to those positions into which the forces it receives from the plate conspire to impel it.

105. All the phenomena observed and described may, as it appears to me, be now comprehended. The fluid may be considered as a pendulum vibrating to and fro under a given impulse; the various circumstances of specific gravity, cohesion, friction, intensity of vibrating force, &c. determining the extent of oscillation, or, what is the same thing, the number of heaps in a given interval. When the number of vibrations in a given time is increased, these heaps are more numerous, because the oscillation, to be more rapid, must occur in a shorter space. The necessity of a certain depth of fluid (73) is evident, and also the reason why, by varying the depth (82), the lateral extent of the heaps is changed. The arrangement of the sand and lycopodium, by the crispations, and the occurrence of the latter at centres of vibration, and only upon surfaces vibrating normally, are all evident consequences. The permanency of the lateral extension of the heaps, when the velocity of the vibrating plate varies, is a very marked effect, and it is probable that the investigation of these phenomena may hereafter importantly facilitate inquiries into the undulations of fluids, their physical qualities, and the transmission of forces through them.

106. As to the origin or determination of crispations, no difficulty can arise; the smallest possible difference in almost any circumstance, at any one part, would, whilst the plate is vibrating, cause an elevation or depression in the fluid there; the smallest atom of dust falling on the surface, or the smallest elevation in the plate, or the smallest particle in the fluid of different specific gravity to the liquid itself, might produce this first effect; this would, by each vibration of the plate, be increased in amount, and also by each vibration extended the breadth of a heap, in at least four directions: so that in less than a second a large surface would be affected, even under the improbable supposition that only one point should at first be disturbed.

107. I have thought it unnecessary to dwell upon the explanation of the circular linear heaps (83. 93. 110) produced on long or circular plates by feeble vibration. They are explicable upon the same principles, account being at the same time taken of the arrangement and proportion of vibrating force in the various parts of the plates.

108. The heaps which constitute crispation (as the word has been used in

this paper) are in form, quality, and motion of their parts, the same with what are called stationary undulations; and if the mercury in a small circular basin be tapped at the middle, stationary undulations, resembling the ring-like heaps (83. 110), will be obtained; or if a rectangular frame be made to beat at equal intervals of time on mercury or water, heaps like those of the crispations, arranged quadrangularly at angles of 45° to the frame, will be produced. These effects are in fact the same with those described, but are produced by a cause differing altogether. The first are the result of two progressing and opposed undulations, the second of four: but the heaps of crispations are produced by the power impressed on the fluid by the vibrating plate; are due to vibrations of that fluid occurring in twice the time of the vibrations of the plate; and have no dependence on progressive undulations, originating laterally, as many of the phenomena described prove. Thus, when the edges were bevelled (72. 110), or covered with cloth, or wet saw-dust, so that waves reaching the side should be destroyed, or when the limits of the water or plates were round (91) or irregular, still the heaps were produced, and their arrangement square. When the round plate (93) was used, regular crispations were still produced, though, as the water extended over the nodal line, and was there perfectly undisturbed, no progressing and opposed undulations could originate to produce them. Vellum stretched over a ring, and rendered concave by the pressure of the exciting rod, produced the same effect.

109. When a plate of tin, rendered very slightly concave, was attached to a lath (69), so as to have equality of vibratory motion in all its parts, and a little dilute alkali (which would wet the surface) put into it, the crispations formed in the middle, but ceased towards the sides, where, though well wetted, there was not depth enough of water, and from whence also no waves could be reflected to produce stationary undulations in the ordinary manner.

110. When a similar arrangement was made with mercury on a concave tin plate, the effects were still more beautiful and convincing. The centre portion was covered with one regular group of quadrangular crispations; at some distance from the centre, and where the mercury was less in depth, these passed into concentric, ring-like heaps, of which there were a great many; and outside of these there was a part wet with mercury, but with too little fluid to give either lines or heaps. Here there could be no reflected waves; or, if that were thought possible, those waves could not have formed both the circular rings and

the square crispation. When this plate was vibrated, the mercury spread in all directions up the side, a natural consequence of the production of powerful oscillations at the middle, which would extend their force laterally, but quite against their being due to the opposition and crossing of waves originating at the sides.

111. A limited depth of fluid is by no means necessary to produce crispations on the surface (73). A circular glass basin about five inches in diameter and four inches deep was attached to a lath (69), filled with water and vibrated, the exciting rod being applied at the side (71). The surface of the water was immediately covered with the most regular crispations, i. e. heaps arranged quadrangularly. On taking out part of the water and filling it up with oil, the oil assumed the same superficies. On putting an inch in depth of mercury under the water, the mercury became crispated. The experiment was finally made with water fourteen inches in depth. Particles at a very moderate depth in the water seemed to have no motion except the general motion of the fluid, and the whole of the lower part of the water may be considered as performing the part of a solid mass upon which the superficial undulating portion reposed. In fact it matters not to the fluid, what is beneath, provided it has sufficient cohesion, is uniform in relation to the surface fluid, and can transmit the vibrations to it in an undisturbed manner *.

112. The beautiful action thus produced at the limits of two immiscible fluids, differing in density or some other circumstances, by which the denser was enabled most readily to accommodate itself to rapid, regular and alternating displacements of its support when that support was horizontal, suggested an inquiry into the probable arrangement of the fluid when the displacements were lateral or even superficial.

113. On arranging the long plate (67. 81) vertically, so that the lower extremity dipped about one third of an inch into water, fig. 27, and causing it to vibrate by applying the rod at ×, or by tapping the plate with the finger, undulations of a peculiar character were observed: those passing from the plate towards the sides of the basin were scarcely visible though the plate vibrated strongly, but in place of such appeared others, in the production of which the mechanical force

Fig. 27.

* I have seen the water in a pail placed in a barrow, and that on the head of an upright cask in a brewer's van passing over stones, exhibit these elevations.

of the vibrating plate exerted upon the fluid was principally employed. These were apparently permanent elevations, at regular intervals, strongest at the plate, projecting directly out from it over the surface of the water, like the teeth of a coarse comb gradually diminishing in height, and extending half or three quarters of an inch in length. These varied in commencing at the glass, or having intervening ridges, or in height, or in length, or in number, or in breaking up into violently agitated pimples and drops, &c. according as the plate dipped more or less into the water, or vibrated more or less violently, or subdivided whilst vibrating into parts, or changed in other circumstances. But when the plate (sixteen or seventeen inches long) dipped about one sixth of an inch, then four of these linear heaps occupied as nearly as possible the same space as four heaps formed with the same plate in the former way (83) and accompanied with the same sound.

114. By fixing a wooden lath (69) perpendicularly downwards in a vice, plates of any size or form could be attached to its lower end and immersed more or less in water; and by varying the immersion of the plate, or the length of the lath, or the place against which the exciting rod (71) was applied, the vibrations could be varied in rapidity to any extent.

115. On using a piece of board at the extremity of the lath, eight inches long and three inches deep, with pieces of tin plate four inches by five, fixed on at the ends in a perpendicular position to prevent lateral disturbance at those parts, very regular and beautiful ridges were obtained of any desired width, fig. 28. These ridges, as before, formed only on the wood, and were parallel to the direction of its vibration. They occurred on each side of the vibrating plane with equal regularity, force and magnitude, but seemed to have no connection, for sometimes they corresponded in position, and at other times not; the one set shifting a little, without the others being displaced.

Fig. 28.

116. It could now be observed that the ridges on either side the vibrating plane consisted of two alternating sets; the one set rising as the other fell. For each fro and to motion of the plane, or one complete vibration, one of the sets appeared, so that in two complete vibrations the cycle of changes was complete. Pieces of cork and lycopodium powder showed that there was no important current setting in the direction of the ridges; towards the heads

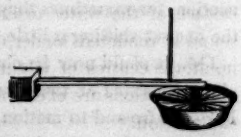
of the ridges pieces of cork oscillated from one ridge towards its neighbour, and back again. The lycopodium sometimes seemed to move on the ridges from the wood, and between them to it; but the motion was irregular, and there was no general current outwards or inwards. There was not so much disturbance as amongst the heaps (90).

117. A very simple arrangement exhibits these ripples beautifully. If an oval or circular pan, fifteen or eighteen inches in diameter, be filled with water, and a piece of lath (69) twelve or fifteen inches long be held in it, edge upwards, so as to bear against the sides of the pan as supporting points, and cut the surface of the water, then on being vibrated horizontally by the glass rod and wet finger, the phenomenon immediately appears with ripples an inch or more in length. When the upper edge of the lath was an inch below the surface, the ripples could be produced. When the vessel had a glass bottom, the luminous figures produced by a light beneath and a screen above, were very beautiful (96). Glass, metal and other plates could thus be easily experimented with.

118. These ripple-like stationary undulations are perfectly analogous as to cause, arrangement and action with the heaps and crispations already explained, i. e. they are the results of that vibrating motion in directions perpendicular to the force applied (105), by which the water can most readily accommodate itself to rapid, regular, and alternating changes in bulk in the immediate neighbourhood of the oscillating parts.

119. From this view of the effect it was evident that similar phenomena would be produced if a substance were made to vibrate in contact with and normally to the surface of a fluid, or indeed in any other direction. A lath was therefore fixed horizontally in a vice by one end, so that the other could vibrate vertically; a cork was cemented to the under surface of the free end, and a basin of water placed beneath with its surface just touching the cork; on vibrating the lath by means of the glass rod and fingers (67), a beautiful and regular star of ridges two, three, or even four inches in length, was formed round the cork, fig. 29. These ridges were more or less numerous according to the number of vibrations, &c. As the water was raised, and more

Fig. 29.



of the cylinder immersed, the ridges diminished in strength, and at last disappeared: when the cylinder of cork just touched the surface, they were most powerfully developed. This is a necessary consequence of the dependence of the ridges upon the portion of water which is vertically displaced and restored at each vibration. When that, being partial in relation to the whole surface, is at or near the surface, the ridges are freely formed in the immediate vicinity; when at a greater depth (being always at the bottom of the cork), the displacement is diffused over a larger mass and surface, each particle moves through less space and with less velocity, and consequently the vibrations must be stronger or the ridges be weaker or disappear altogether. The refraction of a light through this star produces a very beautiful figure on a screen.

120. A heavy tuning-fork vibrating, but not too strongly, if placed with the end of one limb either vertical, inclined, or in any other position, just touching the surface of water, ink, milk, &c. (75), shows the effect very well for a moment. It also shows the ridges on mercury, but the motion and resistance of so dense a body quickly bring the fork to rest. It formed ridges in hot oil, but not in cold oil (76). With cold oil a very inclined fork produced a curious pump-like action, throwing up four streams, easily explained when witnessed, but not so closely connected with the present phenomena as to require more notice here.

121. There is a well known effect of crispation produced when a large glass full of water is made to sound by passing the wet finger round the edges. The glass divides into four vibrating parts opposite to which the crispations are strongest, and there are four nodal points considered in relation to a horizontal section, at equal distances from each other, the finger always touching at one of them. If the vessel is a large glass jar, and soft sounds are produced, the surface of the water exhibits the ridges at the centres of vibration; as the sound is rendered louder, these extend all round the glass, and at last break up at the centres of vibration into irregular crispations, but both the ridges and crispations are effects of the kind already described, and require no further explanation.

122. There are some other effects, one of which I wish here briefly to notice, as connected more or less with the vibratory phenomena that have been described. If, during a strong steady wind, a smooth flat sandy shore, with

enough water on it, either from the receding tide or from the shingles above, to cover it thoroughly, but not to form waves, be observed in a place where the wind is not broken by pits or stones, stationary undulations will be seen over the whole of the wet surface, forming ridges like those already described, and each several inches long. These are not waves of the ordinary kind; they are accurately parallel to the course of the wind; they are of uniform width whatever the extent of surface, varying in width only as the force of the wind and the depth of the stratum of water varies. They may be seen at the windward side of the pools on the sand, but break up so soon as waves appear. If the waves be quelled by putting some oil on the water to windward, these ripples then appear on those parts. They are often seen, but so confused that their nature could not be gathered from such observations, on the pavements, roads, and roofs when sudden gusts of wind occur with rain. The character of these ripples, and their identity with stationary undulations, may be ascertained by exerting the eye and the mind to resolve them into two series of ordinary advancing waves moving directly across the course of the wind in opposite directions. But as such series could not be caused by the wind exerted in a manner similar to that by which ordinary waves are produced, (the direction being entirely opposed to such an idea,) I think the effect is due to the water acquiring an oscillatory condition similar to those described, probably influenced in some way by the elastic nature of the air itself (124) and analogous to the vibration of the strings of the Æolian harp, or even to the vibration of the columns of air in the organ-pipe and other instruments with embouchures.

These ridges were strong enough to arrange the sand beneath where ordinary waves had not been powerful enough to give form to the surface.

123. All the phenomena as yet described are such as take place at the *surfaces* of those fluids in common language considered as inelastic, and in which the elasticity they possess performs no necessary part; nor is it possible that they could be produced within their mass. But on extending the reasoning, it does not seem at all improbable that analogous effects should take place in gases and vapour, their elasticity supplying that condition necessary for vibration which in liquids is found in an abrupt termination of the mass by an unconfined surface.

124. If this be so, then a plate vibrating in the atmosphere may have the air immediately in contact with it separated into numerous portions, forming two alternating sets like the heaps described (95); the one denser, and the other rarer than the ordinary atmosphere; these sets alternating with each other by their alternate expansion and condensation with each vibration of the plate.

125. With the hope of discovering some effect of this kind, a flat circular tin plate had a raised edge of tin three quarters of an inch high fixed on all round, and the plate was then attached to a lath (69), a little lycopodium put on to it, and vibrated powerfully, so that the powder should form a mere cloud in the air, which, in consequence of the raised edge and the equal velocity (70) of all parts of the plate, had no tendency to collect. Immediately it was seen that in place of a uniform cloud it had a misty honeycomb appearance, the whole being in a quivering condition; and on exerting the attention to perceive waves as it were travelling across the cloud in opposite directions, they could be most distinctly traced. This is exactly the appearance that would be produced by a dusty atmosphere lying upon the surface of a plate and divided into a number of alternate portions rapidly expanding and contracting simultaneously.

126. But the spaces were very many times too small to represent the interval through which the air by its elasticity would vibrate laterally once for two vibrations of the plate, in analogy with the phenomena of liquids; and this forms a strong objection to its being an effect of that kind. But it does not seem impossible that the air may have vibrated in subdivisions like a string or a long column of air; and the air itself also being laden with particles of lycopodium would have its motions rendered more sluggish thereby. I have not had time to extend these experiments, but it is probable that a few, well chosen, would decide at once whether these appearances of the particles in the air are due to real lateral vibrations of the atmosphere, or merely to the direct action of the vibrating plate upon the particles.

127. If the atmosphere vibrates laterally in the manner supposed, the effect is probably not limited to the immediate vicinity of the plate, but extends to some distance. The vertical plates intersecting the surface of water and vibrating in a horizontal plane (117) produced ripples proceeding directly out from them five or six inches long; whilst the waves parallel to the vibrating plate

were hardly sensible; and something analogous to this may take place in the atmosphere. If so, it would seem likely that these vibrations occurring conjointly with those producing sound, would have an important influence upon its production and qualities, upon its apparent direction, and many other of its phenomena.

128. Then by analogy these views extend to the undulatory theory of light, and especially to that theory as modified by M. Fresnel. That philosopher, in his profound investigations of the phenomena of light, especially when polarized, has conceived it necessary to admit that the vibrations of the ether take place transversely to the ray of light, or to the direction of the wave causing its phenomena. "In fact we may conceive direct light to be an assemblage, or rather a rapid succession, of an infinity of systems of waves polarized (i. e. vibrating transversely) in all azimuths, and so that there is as much polarized light in any one plane as in a plane perpendicular to it." Herschel says that Fresnel supposes the eye to be affected *only* by such vibrating motions of the ethereal molecules as are performed in planes perpendicular to the direction of the rays. Now the effects in question seem to indicate how the direct vibration of the luminous body may communicate transversal vibration in every azimuth to the molecules of the ether, and so account for that condition of it which is required to explain the phenomena.

129. When the star of ridges formed by a vibrating cylinder (119) upon the surface of water is witnessed instead of the series of circular waves that might be expected, it seems like the instant production of the phenomena of radiation by means of vibratory action. Whether the contiguous rarified and condensed portions which I have supposed in air, gases, vapour and the ether, are arranged radially like the ridges in the experiment just quoted, or whether rare and dense alternate in the direction of the radii as well as laterally, is a question which may perhaps deserve investigation by experiment or calculation.

*Royal Institution,*
*July 30th*, 1831.

## XVIII. *A Table for facilitating the Computations relative to Suspension Bridges.*

*By* Davies Gilbert, *Esq. V.P.R.S.*

Read, May 19, 1831.

THE following Table is supplementary to those accompanying the paper "On the Mathematical Theory of Suspension Bridges," printed in the Philosophical Transactions for 1826. It is deduced from the first Table there given, by the plain operations of common arithmetic; but this admits of a far more ready application than the former, to all cases of practical investigation.

The first column contains the deflections or versed sines of the curve, expressed in fractional parts of the double ordinate or Span. It is therefore $2y$ divided by $x$, and their reciprocals are added under each.

The second column gives the lengths of the chain without alteration from the former Table, except that the double ordinate or span is taken as the unit.

The third column has the tensions of the chain at the middle points or apices of the curve, when the tensions are least; taking the weight of the chain, or that weight augmented by the adjunct weight, or with the adventitious weight also, as unity. The numbers are obtained by dividing $a$ by $2z$.

The fourth column gives the tensions in a similar manner for the extremities of the chain, where they are greatest; and it is made by dividing T by $2z$.

The fifth column gives the angles made by the chains at their extremities with the plane of the horizon, being the complements of those in the former Table.

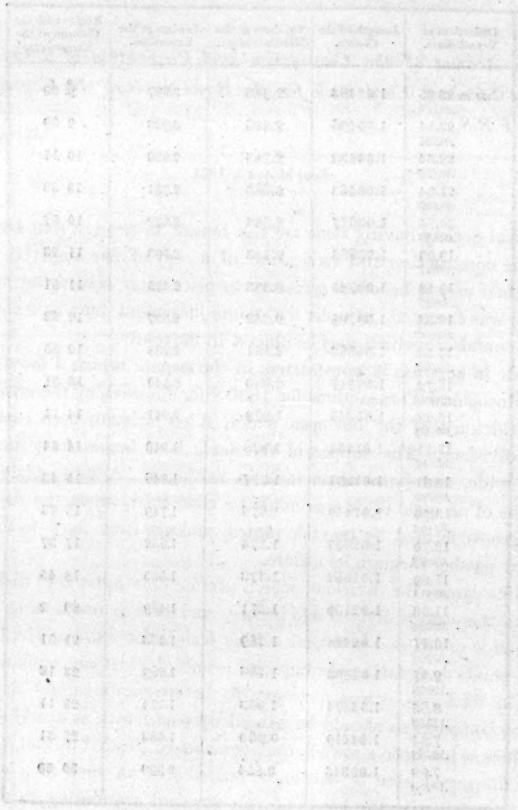
As all these numbers are immediately derived from an existing Table, there would have been much additional trouble, and without any adequate advantage, in making the denominators of the fractions in the first column or their reciprocals (the decimal fractions), to succeed each other by equal differences. And I have thought it unnecessary to extend the Table further in either direction; since no deflection is likely to be so great as a seventh of the span;

and in the event of a short and light bridge being constructed with a deflection less than one part in forty, the columns may be continued with an accuracy quite sufficient, by taking the direct proportion of the denominators for columns three and four, and the inverse proportion for column five. The numbers in column two may be neglected, as not sensibly differing, in such a case, from unity, or they may be found in the Table from which this has been derived.

| Deflections or Versed Sines. | Length of the Chains. | Tensions at the Middle Points. | Tensions at the Extremities. | Angles with the Horizon at the Extremities. |
|---|---|---|---|---|
| One in 39.97 | 1.00166 | 4.992 | 5.017 | ° ′ 5 43 |
| .02502 | | | | |
| 39.17 | 1.00173 | 4.892 | 4.917 | 5 50 |
| .02553 | | | | |
| 38.37 | 1.00181 | 4.791 | 4.817 | 5 58 |
| .02607 | | | | |
| 37.57 | 1.00189 | 4.691 | 4.718 | 6 5 |
| .02662 | | | | |
| 36.76 | 1.00196 | 4.591 | 4.618 | 6 13 |
| .02720 | | | | |
| 35.96 | 1.00206 | 4.491 | 4.519 | 6 21 |
| .02781 | | | | |
| 35.16 | 1.00215 | 4.391 | 4.419 | 6 30 |
| .02844 | | | | |
| 34.36 | 1.00225 | 4.290 | 4.319 | 6 39 |
| .02910 | | | | |
| 33.56 | 1.00236 | 4.190 | 4.220 | 6 48 |
| .02980 | | | | |
| 32.76 | 1.00247 | 4.090 | 4.121 | 6 58 |
| .03053 | | | | |
| 31.96 | 1.00260 | 3.989 | 4.021 | 7 9 |
| .03129 | | | | |
| 31.16 | 1.00273 | 3.889 | 3.921 | 7 22 |
| .03210 | | | | |
| 30.36 | 1.00288 | 3.789 | 3.822 | 7 31 |
| .03294 | | | | |
| 29.56 | 1.00304 | 3.689 | 3.723 | 7 43 |
| .03384 | | | | |
| 28.75 | 1.00322 | 3.588 | 3.623 | 7 56 |
| .03478 | | | | |
| 27.95 | 1.00340 | 3.488 | 3.524 | 8 9 |
| .03577 | | | | |
| 27.15 | 1.00360 | 3.388 | 3.425 | 8 24 |
| .03683 | | | | |
| 26.35 | 1.00383 | 3.287 | 3.325 | 8 39 |
| .03795 | | | | |
| 25.55 | 1.00407 | 3.187 | 3.226 | 8 55 |
| .03914 | | | | |
| 24.75 | 1.00434 | 3.087 | 3.126 | 9 12 |
| .04041 | | | | |

TABLE (Continued).

| Deflections or Versed Sines. | Length of the Chains. | Tensions at the Middle Points. | Tensions at the Extremities. | Angles with the Horizon at the Extremities. |
|---|---|---|---|---|
| | | | | ° ′ |
| One in 23.95 | 1.00463 | 2.986 | 3.027 | 9 30 |
| .04176 | | | | |
| 23.14 | 1.00496 | 2.886 | 2.928 | 9 50 |
| .04321 | | | | |
| 22.34 | 1.00532 | 2.785 | 2.830 | 10 11 |
| .04476 | | | | |
| 21.54 | 1.00568 | 2.685 | 2.731 | 10 33 |
| .04642 | | | | |
| 20.74 | 1.00617 | 2.584 | 2.632 | 10 57 |
| .04823 | | | | |
| 19.93 | 1.00668 | 2.483 | 2.533 | 11 23 |
| .05017 | | | | |
| 19.13 | 1.00725 | 2.383 | 2.435 | 11 51 |
| .05227 | | | | |
| 18.33 | 1.00789 | 2.282 | 2.337 | 12 22 |
| .05456 | | | | |
| 17.52 | 1.00863 | 2.181 | 2.238 | 12 55 |
| .05706 | | | | |
| 16.72 | 1.00947 | 2.080 | 2.140 | 13 31 |
| .05980 | | | | |
| 15.92 | 1.01045 | 1.979 | 2.041 | 14 11 |
| .06282 | | | | |
| 15.11 | 1.01158 | 1.878 | 1.943 | 14 54 |
| .06617 | | | | |
| 14.31 | 1.01291 | 1.777 | 1.846 | 15 43 |
| .06989 | | | | |
| 13.50 | 1.01448 | 1.676 | 1.749 | 16 28 |
| .07406 | | | | |
| 12.70 | 1.01635 | 1.574 | 1.652 | 17 37 |
| .07876 | | | | |
| 11.89 | 1.01862 | 1.473 | 1.555 | 18 45 |
| .08411 | | | | |
| 11.08 | 1.02139 | 1.371 | 1.469 | 20 2 |
| .09024 | | | | |
| 10.27 | 1.02484 | 1.269 | 1.363 | 21 31 |
| .09734 | | | | |
| 9.47 | 1.02893 | 1.166 | 1.269 | 23 12 |
| .10563 | | | | |
| 8.65 | 1.03474 | 1.063 | 1.174 | 25 11 |
| .11559 | | | | |
| 7.84 | 1.04219 | 0.960 | 1.083 | 27 31 |
| .12762 | | | | |
| 7.00 | 1.05343 | 0.854 | 0.990 | 30 20 |
| .14280 | | | | |



## XIX. *An Account of the Construction and Verification of a Copy of the Imperial Standard Yard made for the Royal Society. By Captain* HENRY KATER, *F.R.S.*

Read May 19, 1831.

THE Royal Society having done me the honour to request that I would undertake the construction and verification of a copy of the Imperial Standard Yard for their use, it becomes necessary to place upon record the manner in which this was executed, in order that some judgement may be formed of the degree of confidence which may be placed in the result.

The scale in question is constructed in the manner which I have described in the Philosophical Transactions for 1830 * for diminishing the errors arising from the thickness of the bar upon which it has hitherto been customary to trace the divisions. The support of the scale is of brass, forty inches long, $1\frac{5}{8}$ inches wide, and $\frac{6}{10}$ths of an inch in thickness. A brass plate of seven hundredths of an inch thick was made to slide freely upon the support in a dovetail groove formed by two side plates, and was then fixed to the support by a screw passing through its middle.

This plate carries the divisions, which are fine dots upon gold disks let into the brass; the scale is divided into inches, and there is one inch to the left of zero, which is subdivided into tenths. The scale is the work of Mr. DOLLOND.

As the points designating the Imperial Standard Yard are upon a brass bar one inch in thickness, it was necessary to be extremely careful that the bar during the comparisons should be placed upon a surface as nearly as possible plane; since it has been shown in the paper before alluded to, that a curvature of which the versed sine is only one-hundredth of an inch in a yard would occasion a variation in the length of this standard amounting to nearly five-thousandths of an inch.

* Page 359.

The marble slab formerly used was employed on the present occasion. Its surface was examined by means of a wire, the diameter of which was one-hundredth of an inch, stretched by a bow with a force of about four pounds; but as this wire would suffer a deflexion by its own weight amounting to about four-thousandths of an inch, a wire of two-hundredths of an inch diameter was placed at each extremity of the marble slab, and the wire of the bow resting upon these; the distance of its middle point from the surface of the marble was found to be a little less than two-hundredths of an inch, estimated by passing beneath it a wire of one-hundredth of an inch in diameter. The marble slab being sixty-four inches long, its surface may therefore, perhaps for the extent of a yard, be considered as sufficiently approximating to a plane; and I may here remark that no new adjustment of the slab was found to be necessary, as its position appeared to have undergone no change since my last measurements.

The scale was placed upon the marble slab near the Imperial Standard Yard, and the comparisons were always made about nine o'clock in the morning, in order to ensure as far as possible an equality of temperature in the scale and the Standard Yard. It will be seen that seldom more than three comparisons were taken on the same morning, lest the proximity of the person of the observer might destroy the equality of temperature.

The microscopic apparatus used on the present occasion is that which was employed in the comparison of various British standards of linear measure, an account of which is given in the Philosophical Transactions for 1821, and the mode pursued in making the comparisons was the same as that which I have there detailed. The value of one division of the micrometer is =.0000428742 of an inch. As the microscopes invert, an increase in the readings indicates a corresponding deficiency in the length of the scale.

| Date. 1881. | Imperial Standard Yard. | From 0 to 36 on the Royal Society's Scale. | Difference. | Difference in Inches. |
|---|---|---|---|---|
| | microm. readings. | microm. readings. | div. | inches. |
| April 5 | 46 | 62½ | 16½ | .0007074 |
| | 48½ | 61 | 12½ | .0005358 |
| | 46 | 59½ | 13½ | .0005787 |
| | 43 | 59 | 16 | .0006860 |
| P.M. | 32 | 45½ | 13½ | .0005787 |
| 6 | 36 | 53 | 17 | .0007288 |
| | 38½ | 53 | 14½ | .0006216 |
| | 37 | 54 | 17 | .0007288 |
| | 39 | 52 | 13 | .0005573 |
| 7 | 26½ | 43 | 16½ | .0007074 |
| | 29 | 44 | 15 | .0006431 |
| | 31 | 44 | 13 | .0005573 |
| 8 | 31 | 47 | 16 | .0006860 |
| | 33½ | 47 | 13½ | .0005787 |
| | 34 | 46 | 12 | .0005144 |
| 9 | 51 | 67 | 16 | .0006860 |
| | 51 | 68 | 17 | .0007288 |
| | 52 | 67 | 15 | .0006431 |
| 10 | 43 | 57 | 14 | .0006002 |
| 11 | 35½ | 53½ | 18 | .0007717 |
| | 36 | 54½ | 18½ | .0007931 |
| | 41 | 52 | 11 | .0004716 |
| | 37 | 51 | 14 | .0006002 |
| 12 | 30 | 45 | 15 | .0006431 |
| | 30 | 46 | 16 | .0006860 |
| | 35 | 44½ | 9½ | .0004072 |
| | 30 | 44 | 14 | .0006002 |
| 13 | 39 | 55 | 16 | .0006860 |
| | 37½ | 51½ | 14 | .0006002 |
| | 36 | 50 | 14 | .0006002 |
| 14 | 17 | 29 | 12 | .0005144 |
| | 16 | 27½ | 11½ | .0004930 |
| | 7 | 18 | 11 | .0004716 |
| 15 | 86 | 103½ | 17½ | .0007502 |
| | 91 | 106 | 15 | .0006431 |
| | 92½ | 105 | 12½ | .0005358 |
| | | Mean ..... | | .0006204 |

It will be seen in the above Table that the greatest difference between any one of the thirty-six comparisons and the mean is less than two ten-thousandths of an inch; the distance from 0 to 36 on the Royal Society's scale may therefore be considered as equal to 35.99938 inches of the Imperial Standard Yard.

## XX. *On the Theory of the Elliptic Transcendents.* By James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris, et Soc. Reg. Sc. Gottin., Corresp.*

Read June 9, 1831.

THE branch of the integral calculus which treats of elliptic transcendents originated in the researches of Fagnani, an Italian geometer of eminence. He discovered that two arcs of the periphery of a given ellipse may be determined in many ways, so that their difference shall be equal to an assignable straight line; and he proved that any arc of the lemniscata, like that of a circle, may be multiplied any number of times, or may be subdivided into any number of equal parts, by finite algebraic equations. These are particular results; and it was the discoveries of Euler that enabled geometers to advance to the investigation of the general properties of the elliptic functions. An integral in finite terms deduced by that geometer from an equation between the differentials of two similar transcendent quantities not separately integrable, led immediately to an algebraic equation between the amplitudes of three elliptic functions, of which one is the sum, or the difference, of the other two. This sort of integrals, therefore, could now be added or subtracted in a manner analogous to circular arcs, or logarithms; the amplitude of the sum, or of the difference, being expressed algebraically by means of the amplitudes of the quantities added or subtracted. What Fagnani had accomplished with respect to the arcs of the lemniscata, which are expressed by a particular elliptic integral, Euler extended to all transcendents of the same class. To multiply a function of this kind, or to subdivide it into equal parts, was reduced to solving an algebraic equation. In general, all the properties of the elliptic transcendents, in which the modulus remains unchanged, are deducible from the discoveries of Euler. Landen enlarged our knowledge of this kind of functions, and made a useful addition to analysis, by showing that the arcs of the hyperbola may be reduced, by a proper transformation, to those of the

ellipse. Every part of analysis is indebted to LAGRANGE, who enriched this particular branch with a general method for changing an elliptic function into another having a different modulus, a process which greatly facilitates the numerical calculation of this class of integrals. An elliptic function lies between an arc of the circle on one hand, and a logarithm on the other, approaching indefinitely to the first when the modulus is diminished to zero, and to the second when the modulus is augmented to unit, its other limit. By repeatedly applying the transformation of LAGRANGE, we may compute either a scale of decreasing moduli reducing the integral to a circular arc, or a scale of increasing moduli bringing it continually nearer to a logarithm. The approximation is very elegant and simple, and attains the end proposed with great rapidity.

The discoveries that have been mentioned occurred in the general cultivation of analysis; but LEGENDRE has bestowed much of his attention and study upon this particular branch of the integral calculus. He distributed the elliptic functions in distinct classes, and reduced them to a regular theory. In a Mémoire sur les Transcendantes Elliptiques, published in 1793, and in his Exercices de Calcul Intégral, which appeared in 1817, he has developed many of their properties entirely new; investigated the easiest methods of approximating to their values; computed numerical tables to facilitate their application; and exemplified their use in some interesting problems of geometry and mechanics. In a publication so late as 1825, the author, returning to the same subject, has rendered his theory still more perfect, and made many additions to it which further researches had suggested. In particular we find a new method of making an elliptic function approach as near as we please to a circular arc, or to a logarithm, by a scale of reduction very different from that of which LAGRANGE is the author, the only one before known. This step in advance would unavoidably have conducted to a more extensive theory of this kind of integrals, which, nearly about the same time, was being discovered by the researches of other geometers.

M. ABEL of Christiana, and M. JACOBI of Konigsberg, entirely changed the aspect of this branch of analysis by the extent and importance of their discoveries. The first of these geometers, whom, to the great loss of science, a premature death cut off in the beginning of a career of the highest expectations,

happily conceived the idea of expressing the amplitude of an elliptic function in terms of the function itself. By this procedure the sines and cosines of the amplitudes become periodical quantities like the sines and cosines of circular arcs; and analogy immediately points out many new and useful properties which it would be difficult to deduce by any other mode of investigation. This new way of considering the subject struck out by M. Abel, not only disclosed to him some interesting and original views, but it conducted him to the general and recondite theorems which, without his knowledge, had been previously discovered by the geometer of Konigsberg. M. Jacobi, following in his researches a different method from M. Abel, proved that an elliptic function may be transformed innumerable ways into another similar function to which it bears constantly the same proportion. In the solution of this problem the modulus and the amplitude sought are deduced from the like given quantities, by equations which depend upon the division into an odd number of equal parts of the definite integral, having its amplitude equal to 90°; and, as any odd number may be chosen at pleasure, the number of transformations is unlimited. In consequence of this discovery, an elliptic function can have its modulus augmented or diminished according to an infinite number of different scales. The new process for effecting the same reduction discovered by Legendre in 1825, is only the most simple case of the extensive theorem of M. Jacobi; and, although the older transformation of Lagrange is no part of the same theorem, it bears to it a close resemblance in every respect. Such is the principal addition made to this branch of analysis by M. Jacobi; but the new methods of investigation introduced by him and M. Abel, open a wide field of collateral research, which probably will long continue to furnish matter for exercising the ingenuity of mathematicians.

But it seldom happens that an inventor arrives by the shortest road at the results which he has created, or explains them in the simplest manner. The demonstrations of M. Jacobi require long and complicated calculations; and it can hardly be said that the train of deduction leads naturally to the truths which are proved, or presents all the conclusions which the theory embraces in a connected point of view. The theorem does not comprehend the transformation of Lagrange, which must be separately demonstrated. This is an imperfection of no great moment; but it is always satisfactory to contemplate

a theory in its full extent, and to deduce all the connected truths from the same principles. On a careful examination it will be found that the sines or cosines of the amplitudes used in the transformations are analogous to the sines or cosines of two circular arcs, one of which is a multiple of the other; insomuch that the former quantities are changed into the latter when the modulus is supposed to vanish in the algebraic expressions. We may therefore transfer to the elliptic transcendent the same methods of investigation that succeed in the circle. When this procedure is followed, there is no need to distinguish between an odd and an even number; the demonstrations are shortened; and the difficulties are mostly removed by the close analogy between the two cases. It is in this point of view that the subject is treated in this paper, in which it is proposed to demonstrate the principal theorems without going into the detail of the applications.

1. Elliptic functions of the first kind are of this form *, viz.

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1-h^2\sin^2\psi}};$$

the arcs $\varphi$ and $\psi$ being the amplitudes, and the quantities $k$ and $h$, which are always less than unit, the moduli of the functions. For the sake of abridging, I shall denote the foregoing integrals by $\mathrm{K}(\varphi)$ and $\mathrm{H}(\psi)$, the prefixes K and H having reference to the moduli $k$ and $h$; and, for the definite integral between the amplitudes 0 and $\frac{\pi}{2}$, I shall use indiscriminately either $\mathrm{K}\left(\frac{\pi}{2}\right)$ and $\mathrm{H}\left(\frac{\pi}{2}\right)$, or, more simply, K and H.

The general equation to be investigated is the following,

$$\int \frac{d\psi}{\sqrt{1-h^2\sin^2\psi}} = \beta \int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad (1)$$

$\beta$ being a constant quantity equal to the first ratio of the nascent arcs $\psi$ and $\varphi$.

* In what follows, the terms 'elliptic functions' and 'elliptic transcendents' are to be understood as applying to those of the first kind only, which alone are treated of.

If we admit that this is a possible equation, and suppose that when $\psi$ is successively equal to the arcs of the series,

$$0, \quad \frac{\pi}{2}, \quad 2\frac{\pi}{2}, \quad 3\frac{\pi}{2}, \quad \&c.,$$

$\phi$ attains the respective values,

$$0, \lambda_1, \lambda_2, \lambda_3, \&c.;$$

we shall have,

$$H = \beta K(\lambda_1), 2H = \beta K(\lambda_2), 3H = \beta K(\lambda_3), \&c.;$$

and consequently,

$$K(\lambda_2) = 2K(\lambda_1), \quad K(\lambda_3) = 3K(\lambda_1), \&c.$$

Thus the arcs $\lambda_2$, $\lambda_3$, &c. are the amplitudes of the multiples of the function $K(\lambda_1)$, which itself remains indeterminate. We may therefore suppose $p \times K(\lambda_1) = K\left(\frac{\pi}{2}\right)$, $p$ representing any integer number; and, in consequence, we shall have

$$K(\lambda_1) = \frac{1}{p}K, \quad K(\lambda_2) = \frac{2}{p}K, \quad \ldots \quad K(\lambda_m) = \frac{m}{p}K.$$

Any proposed number being assumed for $p$, we may determine the amplitudes $\lambda_1$, $\lambda_2$, $\lambda_3$, &c. by the theory for the multiplication and subdivision of elliptic functions: but as the equations to be solved are complicated and impracticable, the arcs $\lambda_1$, $\lambda_2$, &c. may be treated as known quantities without any attempt to compute them.

An elliptic function becomes equal to the arc of its amplitude, when the modulus vanishes: and in this case the arcs $\lambda_1$, $\lambda_2$, $\lambda_3$, &c. are obtained by the subdivision of the quadrant of the circle, and are respectively equal to $\frac{1}{p} \cdot \frac{\pi}{2}$, $\frac{2}{p} \cdot \frac{\pi}{2}$, $\frac{3}{p} \cdot \frac{\pi}{2}$, &c.

Having made these observations, we shall for the present dismiss all consideration of the equation to be demonstrated, and turn our attention to investigate two variable arcs $\psi$ and $\phi$, such that the first shall have the successive values,

$$0, \quad \frac{\pi}{2}, \quad 2 \times \frac{\pi}{2}, \quad 3 \times \frac{\pi}{2}, \&c.$$

when the second becomes respectively equal to the several known amplitudes,

$$0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \&c.$$

2. As we shall have occasion to refer to the formulas for the addition and subtraction of elliptic functions, it will be convenient to premise them.

Let $a$ and $b$ represent any two amplitudes, and put

$$\mathrm{K}(a) + \mathrm{K}(b) = \mathrm{K}(s)$$
$$\mathrm{K}(a) - \mathrm{K}(b) = \mathrm{K}(\sigma):$$

then, according to the formulas of EULER *,

$$\sin s = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} + \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\sin \sigma = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} - \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}.$$

From these we immediately deduce,

$$\sin s \sin \sigma = \frac{\sin^2 a - \sin^2 b}{1 - k^2 \sin^2 a \sin^2 b} \dagger. \qquad (A)$$

It may be observed that if $a = \lambda_m$, $b = \lambda_n$; then $s = \lambda_{m+n}$, $\sigma = \lambda_{m-n}$: for it is obvious that

$$\mathrm{K}(\lambda_m) + \mathrm{K}(\lambda_n) = (m + n)\,\mathrm{K}(\lambda_1) = \mathrm{K}(\lambda_{m+n})$$
$$\mathrm{K}(\lambda_m) - \mathrm{K}(\lambda_n) = (m - n)\,\mathrm{K}(\lambda_1) = \mathrm{K}(\lambda_{m-n})$$

3. In order to avail ourselves of the analogy between the elliptic functions and the arcs of a circle, we must take that view of the matter first suggested by M. ABEL. Let

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \mathrm{K}(\varphi);$$

then, as $u$ is a variable quantity depending upon the amplitude $\varphi$, reciprocally this latter quantity will depend upon the first; which dependance we shall express in this manner,

$$\varphi = \text{amplitude of } u = \mathrm{A}\,u,$$
$$\sin \varphi = \sin \mathrm{A}\,u.$$

* Traité des Fonctions Elliptiques, tom. i. p. 22.

† This equation is called by M. ABEL "la propriété fondamentale."

For the sake of abridging, let $\omega = \frac{1}{p}\,\mathrm{K}$; so that $\lambda_1, \lambda_2, \lambda_3$, &c. will be the respective amplitudes of $\omega$, $2\omega$, $3\omega$, &c.; and $\lambda_p$ = amplitude of $p\omega$ = amp. of $\mathrm{K} = 90°$; and $\lambda_{2p}$ = amp. of $2p\omega$ = amp. of $2\,\mathrm{K} = 180°$. From the nature of the integral, it follows that when $u$ receives an addition equal to $2p\omega$ or $2\,\mathrm{K}$, the amplitude of $u$ will be increased by 180°.

To the indeterminate quantity $u$ let there be added the several even multiples of $\omega$ less than $2p\omega$; and let us consider the sines of the amplitudes of the functions so formed, viz.

$$\sin \mathrm{A}\, u,\ \sin \mathrm{A}\,(u + 2\omega),\ \sin \mathrm{A}\,(u + 4\omega),\ \ldots\ \sin \mathrm{A}\,(u + 2p\omega - 2\omega):$$

in this series, if we substitute in place of $u$, the successive quantities $u + 2\omega$, $u + 4\omega$, $u + 6\omega$, &c., the same sines will constantly recur in periodical order, abstracting from the change of sign when an amplitude becomes greater than 180°, or than a multiple of 180°. Thus, if we put $u + 2\omega$ in place of $u$, the second term of the foregoing series will stand first, and the last term will be $\sin \mathrm{A}\,(u + 2p\omega) = -\sin \mathrm{A}\,(u)$. In like manner, if $u + 4\omega$ be substituted for $u$, the third term of the series will stand first, and the two last terms will be $-\sin \mathrm{A}\, u$, $-\sin \mathrm{A}\,(u + 2\omega)$; and so on.

Let us now put

$$\gamma = \sin \mathrm{A}\,\omega \times \sin \mathrm{A}\,(3\omega) \times \sin \mathrm{A}\,(5\omega)\ \ldots\ \times \sin \mathrm{A}\,(2p\omega - \omega)$$

or, which is the same thing,

$$\gamma = \sin \lambda_1 \times \sin \lambda_3 \times \sin \lambda_5\ \ldots\ \times \sin \lambda_{2p-1};$$

and further, let us assume,

$$y = \frac{\sin \mathrm{A}\, u \times \sin \mathrm{A}\,(u + 2\omega) \times \sin \mathrm{A}\,(u + 4\omega)\ \ldots\ \times \sin \mathrm{A}\,(u + 2p\omega - 2\omega)}{\gamma}. \qquad \text{(B)}$$

In this expression, if we substitute for $u$, the several odd multiples of $\omega$ in succession, viz.

$\omega$, $3\omega$, $5\omega$, $7\omega$, &c.

it follows, from what has been said, that the products in the numerator will always be the same, and equal to the denominator, but that their signs will change alternately as the successive quantities are substituted. Thus, when

any odd multiple $(2n+1)\omega$ is substituted for $u$ in the expression (B), the value of $y$ is always $+1$ or $-1$, according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of $\omega$.

On the other hand, when $u$ is zero, or equal to $2n\omega$ any even multiple of $\omega$, we shall have $y=0$, one of the factors of the numerator necessarily vanishing; for in a sequence of the even multiples of $\omega$, of which the number is $p$, there must be one equal to $2p\omega$, or to a multiple of $2p\omega$; and therefore when $u=2n\omega$, one of the factors must be the sine of an amplitude equal to 180° or to a multiple of 180°.

Further, let $\omega - z$ be substituted for $u$ in the expression (B), $z$ being less than $\omega$; then,

$$y=\frac{\sin A(\omega-z)\sin A(3\omega-z)\ldots.\sin A(2p\omega-\omega-z)}{\gamma}.$$

Now, in the numerator, the partial products, of the first and last factors, of the second and last but one, and so on, are as follows:

$$\sin A(\omega-z)\sin A(2p\omega-\omega-z)=\sin A(\omega-z)\sin A(\omega+z),$$

$$\sin A(3\omega-z)\sin A(2p\omega-3\omega-z)=\sin A(3\omega-z)\sin A(3\omega+z),\text{ \&c.}$$

to which we must add the single factor $\sin A(p\omega-z)$, when $p$ is an odd number. All the partial products, it will be observed, have the same value whether $z$ be positive or negative; and they are all greatest, when $z=0$, as will readily appear from what is proved in § 2. Wherefore $y$ has the same value and the same sign, when $u$ is at equal distances from the limits 0 and $2\omega$; and it attains its greatest magnitude, equal to 1, when $u=\omega$. And, if we substitute $(2n+1)\omega - z$ for $u$, this substitution will not change the foregoing factors, but only their order, and the sign of their product, which sign, while $u$ is contained between the limits $2n\omega$ and $2n\omega+2\omega$, will be $+$ or $-$, according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of $\omega$.

We may now conclude, from what has been proved, that $y$, in the expression (B), represents the sine of an arc $\psi$, which increases from zero with the elliptic function $u$, and coincides with the successive terms of the series,

$$0,\ \frac{\pi}{2},\ 2\frac{\pi}{2},\ 3\frac{\pi}{2},\ \text{\&c. ad infinitum,}$$

at the same time that $u$ attains the values,

$$0,\ \omega,\ 2\omega,\ 3\omega,\ \&c.\ \text{ad infinitum},$$

or, when the amplitude of $u$ becomes equal to the several known arcs,

$$0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \&c.\ \text{ad infinitum}:$$

and further, that there is but one value of $y$, or of $\sin\psi$, between the two consecutive terms $m \times \frac{\pi}{2}$ and $(m+1) \times \frac{\pi}{2}$, for any given value of $u$ between the limits $m\omega$ and $(m+1)\omega$, or for any given amplitude between the arcs $\lambda_m$ and $\lambda_{m+1}$.

4. In what has been proved, $p$ may be either an odd or an even number; but we must now distinguish between the two cases, in like manner as it is necessary to do when we investigate the sine of a multiple of a circular arc. Representing the amplitude of $u$ by $\varphi$, we shall have, $u = \mathrm{K}(\varphi)$, and $\sin\phi = \sin \mathrm{A}\, u$. When $p$ is odd, there will be an even number of factors after the first in the numerator of the expression of $y$ or $\sin\psi$; and any one of these, as $\sin \mathrm{A}\,(u + 2n\omega)$, will have another, namely, $\sin \mathrm{A}\,(u + 2p\omega - 2n\omega) = \sin \mathrm{A}\,(2n\omega - u)$, answering to it; and the product of this pair of factors, viz. $\sin \mathrm{A}\,(u + 2n\omega) \times \sin \mathrm{A}\,(2n\omega - u)$, will be found by the formula (A) of § 2, observing that $\sin a = \sin \mathrm{A}\,(2n\omega) = \sin\lambda_{2n}$, $\sin b = \sin \mathrm{A}\, u = \sin\varphi$, $\sin s = \sin \mathrm{A}\,(2n\omega + u)$, $\sin\sigma = \sin(2n\omega - u)$:
thus we have,

$$\sin \mathrm{A}\,(u + 2n\omega) \sin \mathrm{A}\,(u + 2p\omega - 2n\omega) = \frac{\sin^2\lambda_{2n} - \sin^2\varphi}{1 - k^2 \sin^2\lambda_{2n} \sin^2\phi}.$$

Wherefore, by taking in all the factors and writing $z$ for $\sin\varphi$, we shall obtain,

$$\sin\psi = \frac{z}{\gamma} \cdot \frac{\sin^2\lambda_2 - z^2}{1 - k^2 z^2 \sin^2\lambda_2} \cdot \frac{\sin^2\lambda_4 - z^2}{1 - k^2 z^2 \sin^2\lambda_4} \cdots \frac{\sin^2\lambda_{p-1} - z^2}{1 - k^2 z^2 \sin^2\lambda_{p-1}}.$$

The expression of $\gamma$, viz.

$$\gamma = \sin\lambda_1 \,.\, \sin\lambda_3 \,.\, \sin\lambda_5 \ldots \sin\lambda_{2p-1},$$

may be written in this form,

$$\gamma = \sin^2\lambda_1 \,.\, \sin^2\lambda_3 \,.\, \sin^2\lambda_5 \ldots . \sin^2\lambda_{p-1},$$

omitting the factor $\sin\lambda_p = 1$: wherefore, if we assume,

$$\beta = \frac{\sin^2\lambda_2 \cdot \sin^2\lambda_4 \cdot \sin^2\lambda_6 \ldots \sin^2\lambda_{p-1}}{\sin^2\lambda_1 \cdot \sin^2\lambda_3 \cdot \sin^2\lambda_5 \ldots \sin\lambda_{p-2}},$$

we shall have,

$p$ being an odd number,

$$\sin\psi = \beta z \cdot \frac{1-\frac{z^2}{\sin^2\lambda_2}}{1-k^2z^2\sin^2\lambda_2} \cdot \frac{1-\frac{z^2}{\sin^2\lambda_4}}{1-k^2z^2\sin^2\lambda_4} \cdots \frac{1-\frac{z^2}{\sin^2\lambda_{p-1}}}{1-k^2z^2\sin^2\lambda_{p-1}}. \qquad (2)$$

When $p$ is an even number, if we leave out the first factor in the numerator of the expression of $y$ or $\sin\psi$, there will remain an odd number of factors, that which occupies the middle place, being $\sin A\,(u+p\omega)$: and any factor, as $\sin A\,(u+2n\omega)$, between the first and the middle one, will have another, viz. $\sin A\,(u+2p\omega-2n\omega)$, corresponding to it after the middle one; and the product of this pair of factors will be obtained as before, viz.

$$\sin A\,(u+2n\omega)\sin A\,(u+2p\omega-2n\omega) = \frac{\sin^2\lambda_{2n}-\sin^2\varphi}{1-k^2\sin^2\lambda_{2n}\sin^2\varphi}.$$

With regard to the middle factor, we shall have, in the formulas of § 2, $\sin a = \sin A\,(p\omega) = \sin 90^\circ$, $\sin b = \sin A\,u = \sin\varphi$, $\sin s = \sin A\,(u+p\omega)$; and

$$\sin A\,(u+p\omega) = \frac{\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

Wherefore, by proceeding as before, we shall have,

$p$ being an even number,

$$\sin\psi = \frac{\beta z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} \cdot \frac{1-\frac{z^2}{\sin^2\lambda_2}}{1-k^2z^2\sin^2\lambda_2} \cdot \frac{1-\frac{z^2}{\sin^2\lambda_4}}{1-k^2z^2\sin^2\lambda_4} \cdots \frac{1-\frac{z^2}{\sin^2\lambda_{p-2}}}{1-k^2z^2\sin^2\lambda_{p-2}} \qquad (3)$$

$$\beta = \frac{\sin^2\lambda_2 \cdot \sin^2\lambda_4 \cdot \sin^2\lambda_6 \ldots \sin^2\lambda_{p-2}}{\sin^2\lambda_1 \cdot \sin^2\lambda_3 \cdot \sin^2\lambda_5 \ldots \sin^2\lambda_{p-1}},$$

In both the formulas (2) and (3), it is obvious that $\beta$ is the quotient of the product of the sines of all the even amplitudes, $\lambda_2$, $\lambda_4$, &c. between the limits 0 and 180°, divided by the product of the sines of all the odd amplitudes, $\lambda_1$, $\lambda_3$, &c. contained between the same limits. The general expression of $\beta$, common to the two cases, is therefore as follows,

$$\beta = \frac{\sin\lambda_2 \cdot \sin\lambda_4 \cdot \sin\lambda_6 \ldots \sin\lambda_{2p-2}}{\sin\lambda_1 \cdot \sin\lambda_3 \cdot \sin\lambda_5 \ldots \sin\lambda_{2p-1}}. \qquad (4)$$

In the formula (2) let P and R stand for the products of the binomials in the numerator and denominator; then,

$$\sin\psi = \frac{\beta z P}{R};$$

and,

$$\cos^2\psi = \frac{R^2 - \beta^2 z^2 P^2}{R^2}.$$

The numerator of this expression is a rational function of $z^2$, and it will vanish whenever $\cos^2\psi = 0$, or $\sin^2\psi = 1$, that is, when $z^2$ is equal to $\sin^2\lambda_{2n+1}$, $2n+1$ being any odd number less than $2p$. Suppose that $2n+1$ is any odd number less than $p$, the numerator of the value of $\cos^2\psi$ will be divisible by $\left(1 - \frac{z^2}{\sin^2\lambda_{2n+1}}\right)$, and also by $\left(1 - \frac{z^2}{\sin^2\lambda_{2p-2n-1}}\right)$; and as these binomials are equal, it will be divisible by their product $\left(1 - \frac{z^2}{\sin^2\lambda_{2n+1}}\right)^2$; and, $p$ being itself an odd number, to the double divisors there must be added the single one $\left(1 - \frac{z^2}{\sin^2\lambda_p}\right) = 1 - z^2$. The numerator is therefore divisible by the product,

$$(1-z^2)\cdot\left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2\cdot\left(1 - \frac{z^2}{\sin^2\lambda_3}\right)^2\cdot\ldots\left(1 - \frac{z^2}{\sin^2\lambda_{p-2}}\right)^2:$$

and, as the two expressions have the same absolute term and the same dimensions, they must be identical. Wherefore we have,

$p$ being an odd number,

$$\cos\psi = \sqrt{1-z^2}\cdot\frac{1 - \frac{z^2}{\sin^2\lambda_1}}{1 - k^2z^2\sin^2\lambda_2}\cdot\frac{1 - \frac{z^2}{\sin^2\lambda_3}}{1 - k^2z^2\sin^2\lambda_4}\cdots\frac{1 - \frac{z^2}{\sin^2\lambda_{p-2}}}{1 - k^2z^2\sin^2\lambda_{p-1}}. \qquad (5)$$

In like manner, if P and R represent the rational binomial products in the numerator and denominator of the formula (3), we shall have

$$\sin\psi = \frac{\beta z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}}\times\frac{P}{R};$$

and

$$\cos^2\psi = \frac{(1-k^2z^2)R^2 - \beta^2z^2(1-z^2)P^2}{(1-k^2z^2)R^2}.$$

Proceeding as before, it will appear that the numerator of this expression is

divisible by the double divisor $\left(1-\frac{x^2}{\sin^2\lambda_{2n+1}}\right)^2$, $2n+1$ being any odd number less than $p$; and in this case when $p$ is an even number, all the divisors are double. Wherefore the product

$$\left(1-\frac{x^2}{\sin^2\lambda_1}\right)^2\cdot\left(1-\frac{x^2}{\sin^2\lambda_3}\right)^2\cdot\left(1-\frac{x^2}{\sin^2\lambda_5}\right)^2\cdots\left(1-\frac{x^2}{\sin^2\lambda_{p-1}}\right)^2$$

will divide the numerator of the value of $\cos^2\psi$; and it will be identical to it, because both the expressions have the same dimensions. Thus we obtain,

$p$ being an even number,

$$\cos\psi=\frac{1-\frac{x^2}{\sin^2\lambda_1}}{\sqrt{1-k^2x^2}}\cdot\frac{1-\frac{x^2}{\sin^2\lambda_3}}{1-k^2x^2\sin^2\lambda_2}\cdot\cdots\cdots\frac{1-\frac{x^2}{\sin^2\lambda_{p-1}}}{1-k^2x^2\sin^2\lambda_{p-2}}. \qquad (6)$$

From the equations (2) and (5) we deduce,

$$\tan\psi=\frac{\beta x}{\sqrt{1-x^2}}\cdot\frac{1-\frac{x^2}{\sin^2\lambda_2}}{1-\frac{x^2}{\sin^2\lambda_1}}\cdot\frac{1-\frac{x^2}{\sin^2\lambda_4}}{1-\frac{x^2}{\sin^2\lambda_3}}\cdot\cdots\frac{1-\frac{x^2}{\sin^2\lambda_{p-1}}}{1-\frac{x^2}{\sin^2\lambda_{p-2}}}$$

but it will readily appear that

$$\frac{1-\frac{\sin^2\varphi}{\sin^2\lambda_{2n}}}{1-\frac{\sin^2\varphi}{\sin^2\lambda_{2n+1}}}=\frac{1-\frac{\tan^2\varphi}{\tan^2\lambda_{2n}}}{1-\frac{\tan^2\varphi}{\tan^2\lambda_{2n+1}}};$$

wherefore we obtain,

$p$ being an odd number,

$$\tan\psi=\beta\tan\varphi\times\frac{1-\frac{\tan^2\varphi}{\tan^2\lambda_2}}{1-\frac{\tan^2\varphi}{\tan^2\lambda_1}}\cdot\frac{1-\frac{\tan^2\varphi}{\tan^2\lambda_4}}{1-\frac{\tan^2\varphi}{\tan^2\lambda_3}}\cdot\cdots\frac{1-\frac{\tan^2\varphi}{\tan^2\lambda_{p-1}}}{1-\frac{\tan^2\varphi}{\tan^2\lambda_{p-2}}}. \qquad (7)$$

And in a similar manner we deduce from the equations (3) and (6),

$p$ being an even number,

$$\tan\psi=\frac{\beta\tan\varphi}{1-\frac{\tan^2\phi}{\tan^2\lambda_1}}\cdot\frac{1-\frac{\tan^2\phi}{\tan^2\lambda_2}}{1-\frac{\tan^2\varphi}{\tan^2\lambda_3}}\cdot\frac{1-\frac{\tan^2\phi}{\tan^2\lambda_4}}{1-\frac{\tan^2\phi}{\tan^2\lambda_5}}\cdot\cdots\frac{1-\frac{\tan^2\phi}{\tan^2\lambda_{p-2}}}{1-\frac{\tan^2\phi}{\tan^2\lambda_{p-1}}}. \qquad (8)$$

The formulas (2), (5), (7), in which $p$ is an odd number, are those used in

the theorems of M. Jacobi; the other three, (3), (6), and (8), have been added here. All the formulas will be true in the circle, if we make $k = 0$, and derive the arcs $\lambda_1$, $\lambda_2$, &c. from the subdivision of the quadrant, in like manner as they have been obtained from the subdivision of the definite integral $k\left(\frac{\pi}{2}\right)$. The coefficient $\beta$ is the expression of the first ratio of the nascent arcs $\psi$ and $\varphi$; and it is equal to $p$ in the circle.

All the formulas are, however, imperfect in one respect: they all suppose that the amplitudes $\lambda_1$, $\lambda_2$, &c., derived from the subdivision of the definite integral $k\left(\frac{\pi}{2}\right)$, are known. By means of these amplitudes, the general solution of the problem has been deduced from a particular case: but the formulas cannot be considered as complete till all the coefficients have been expressed in functions of the modulus $k$; and, with respect to this point, the researches of analysts have not yet been entirely successful.

5. Having now investigated the relation between the arcs $\psi$ and $\varphi$, we have next to demonstrate that the equation (1) is true when these amplitudes are substituted in it, and a proper value is assigned to the indeterminate modulus $h$; but this requires some preparation, in order to avoid complicated operations.

First, $p$ being an odd number, we have,

$$\sin\psi = \frac{\beta \cdot z\,\mathrm{P}}{\mathrm{R}}, \quad \cos\psi = \frac{\sqrt{1-z^2}\cdot \mathrm{Q}}{\mathrm{R}};$$

R, P, Q, representing the rational binomial products in the denominators and numerators of the equations (2) and (5): we therefore obtain,

$$\mathrm{R}^2 = \beta^2 z^2 \mathrm{P}^2 + (1 - z^2)\,\mathrm{Q}^2.$$

This equation has been found on the supposition that $z$ is less than 1; but, as it contains no radical quantities, it will be true for all values of $z$. We may therefore substitute $\frac{1}{kz}$ for $z$; and, in the resulting equation, the symbol $z$ will still represent a quantity unrestricted in its value. Now, the substitution of $\frac{1}{kz}$ for $z$ being made, we shall obtain,

$$\mathrm{R}^2 - \beta^2 h^2 z^2 \mathrm{P}^2 = (1 - h^2 z^2)\,\mathrm{R}'^2,$$

in which expression R and P denote the same functions of $z$ as before, and the values of the new symbols $h$ and R′ are as follows,

$$h = k^p \sin^4\lambda_1 \cdot \sin^4\lambda_3 \cdot \sin^4\lambda_5 \ldots \sin^4\lambda_{p-2},$$

$$R' = (1 - k^2 z^2 \sin^2\lambda_1)(1 - k^2 z^2 \sin^2\lambda_3) \ldots (1 - k^2 z^2 \sin^2\lambda_{p-2}).$$

We thus have

$$\left.\begin{aligned} \sqrt{R^2 - \beta^2 z^2 P^2} &= \sqrt{1 - z^2} \cdot Q \\ \sqrt{R^2 - \beta^2 h^2 z^2 P^2} &= \sqrt{1 - k^2 z^2} \cdot R'. \end{aligned}\right\} \quad (C)$$

Secondly, when $p$ is an even number, R, P, Q will stand for the rational binomial products in the denominators and numerators of the equations (3) and (6): thus

$$\sin\psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}, \cos\psi = \frac{1}{\sqrt{1 - k^2 z^2}} \cdot \frac{Q}{R};$$

consequently

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And if in this equation we substitute $\frac{1}{kz}$ in place of $z$, we shall obtain this result,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + R'^2$$

R and P representing the same functions of $z$ as before, and the new symbols $h$ and R′ standing for these values,

$$h = k^p \cdot \sin^4\lambda_1 \cdot \sin^4\lambda_3 \cdot \sin^4\lambda_5 \ldots \sin^4\lambda_{p-1}$$

$$R' = (1 - k^2 z^2 \sin^2\lambda_1)(1 - k^2 z^2 \sin^2\lambda_3) \ldots (1 - k^2 z^2 \sin^2\lambda_{p-1}.$$

From what has been proved we now have

$$\left.\begin{aligned} \sqrt{(1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2} &= Q, \\ \sqrt{(1 - k^2 z^2) R^2 - \beta^2 h^2 z^2 (1 - z^2) P^2} &= R' \end{aligned}\right\} \quad (D)$$

To these formulas we must add the following principle of analysis, on which the demonstration we have in view mainly turns. Let V and U denote rational functions of $z$: we shall have this identical equation,

$$\frac{d.\left(\frac{V}{U}\right)}{dz}U^2 = \frac{1}{a}\left\{(V+aU)\frac{d.(V-aU)}{dz} - (V-aU)\frac{d.(V+aU)}{dz}\right\}:$$

from which it follows, that every double binomial factor either of $V+aU$, or of $V-aU$, is a simple binomial factor of $\frac{d.\left(\frac{V}{U}\right)}{dz}U^2$; and further, if $V+aU$ and $V-aU$ have no common divisor, that every double binomial factor of $(V+aU)\times(V-aU) = V^2 - a^2U^2$, is a simple binomial factor of $\frac{d.\left(\frac{V}{U}\right)}{dz}.U^2$.

6. The differential of the equation (1) may now be readily demonstrated, supposing that $\beta$ has the value investigated in § 4, and $h$, the value assigned to it in § 5. And first when $p$ is an odd number, we obtain from the equation (2),

$$\sin\psi = \frac{\beta.zP}{R}, \quad z = \sin\phi:$$

and with these values the equation (1) will become

$$\frac{\frac{R^2}{dz}d.\left(\frac{zP}{R}\right)}{\sqrt{(R^2-\beta^2z^2P^2)(R^2-\beta^2h^2z^2P^2)}} = \frac{1}{\sqrt{1-z^2.1-h^2z^2}}:$$

and, on account of the formulas (C),

$$\frac{\frac{R^2}{dz}d.\left(\frac{zP}{R}\right)}{Q.R'} = 1.$$

Now it is evident that $R+\beta.zP$ and $R-\beta.zP$, have no common divisor: for, as R contains only the even powers of $z$, and $zP$ only the odd powers, if $1+cz$ be a factor of $R+\beta.zP$, $1-cz$ will necessarily be a factor of $R-\beta.zP$. Wherefore, according to what has been proved above, every double binomial factor of $R^2-\beta^2z^2P^2$, that is, every factor of Q, will be a factor of the function in the numerator of the left side of the last equation. In the very same manner it is proved that every double binomial factor of $R^2-\beta^2h^2z^2P^2$, that is, every factor of R′, will be a factor of the same function.

Wherefore the numerator of the left side of the last equation is divisible by the product $Q \times R'$ in the denominator; and, as both the expressions have the same dimensions and the same absolute term, they are identical; which verifies the equation. Wherefore the equation (1) is demonstrated when $p$ is an odd number.

Secondly, when $p$ is an even number, we have by equation (3),

$$\sin \psi = \frac{\beta z \sqrt{1-z^2}}{\sqrt{1-k^2 z^2}} \cdot \frac{P}{R}, \sin \phi = z:$$

and the differential of equation (1) will become by substitution,

$$\frac{\frac{(1-k^2z^2)\,R^2}{dz} \cdot d \cdot \left(\frac{z\sqrt{1-z^2} \cdot P}{\sqrt{1-k^2z^2} \cdot R}\right)}{\sqrt{\left((1-k^2z^2)\,R^2 - \beta^2 z^2 (1-z^2)\,P^2\right)\left((1-k^2z^2)\,R^2 - \beta^2 k^2 z^2 (1-z^2)\,P^2\right)}}$$

$$= \frac{1}{\sqrt{1-z^2} \cdot 1 - k^2 z^2}: \text{ and, on account of the formulas (D),}$$

$$\frac{\frac{(1-k^2z^2)\,R^2}{dz}\, d \cdot \left(\frac{z\sqrt{1-z^2} \cdot P}{\sqrt{1-k^2z^2} \cdot R}\right)}{Q \cdot R'} = \frac{1}{\sqrt{1-z^2} \cdot 1 - k^2 z^2}.$$

It will be proved, by the like reasoning as before, that the numerator of the left side of this equation is divisible by the product in the denominator. Now if we perform the differentiation indicated, we shall find,

$$S = (1 - 2z^2 + k^2 z^4)\,P\,R + z\,(1-z^2)\,(1-k^2 z^2)\,R^2 \frac{d\left(\frac{P}{R}\right)}{dz},$$

$$\frac{(1-k^2z^2)\,R^2}{dz}\, d \cdot \left(\frac{z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} \cdot \frac{P}{R}\right) = \frac{S}{\sqrt{1-z^2} \cdot 1 - k^2 z^2}:$$

and it is evident that all the rational factors of the left side of this last formula, and consequently all the factors of $Q \times R'$, will be factors of S. By substitution the differential equation (1) will now become

$$\frac{S}{Q \times R'} = 1,$$

which is manifestly verified: for, as $Q \times R'$ divides S, and the two expressions

have the same dimensions and the same absolute term, they are identical. The equation (1) is therefore demonstrated when $p$ is an even number.

7. The transformation expressed by the equation,

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1-h^2\sin^2\psi}} = \beta \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

has now been demonstrated for any number whether odd or even, the constant $\beta$ being determined by the formula (4), and the modulus $h$ by the special formulas in § 5, or, generally without distinguishing whether $p$ is odd or even, by this formula,

$$h = k^p \cdot (\sin\lambda_1 \sin\lambda_3 \sin\lambda_5 \ldots \sin\lambda_{2p-1})^2, \qquad (9)$$

the sines multiplied together being those of all the odd amplitudes less than 180°. The relation between the variable amplitudes $\psi$ and $\varphi$ is expressed by the several equations in § 4.

In order to render the solution of the problem more complete, it may be proper to add a useful method of computing the amplitude $\psi$.

In § 5 we have obtained this equation,

$$\mathrm{R}^2 = \beta^2 x^2 \mathrm{P}^2 + (1-x^2)\,\mathrm{Q}^2.$$

And, if we represent by N and M the products of the binomials in the numerator and denominator of the equation (7), we shall have

$$\tan\psi = \frac{\beta \tan\varphi\, \mathrm{N}}{\mathrm{M}}.$$

Let $x = \tan\varphi$, then $x^2 = \sin^2\varphi = \frac{x^2}{1+x^2}$; and, observing that

$$1 - \frac{x^2}{\sin^2\lambda_n} = \frac{1 - \frac{x^2}{\tan^2\lambda_n}}{(1+x^2)},$$

it will readily appear that

$$\beta x \mathrm{P} = \frac{\beta x \mathrm{N}}{(1+x^2)^{\frac{p}{2}}}, \qquad \sqrt{1-x^2}\,.\,\mathrm{Q} = \frac{\mathrm{M}}{(1+x^2)^{\frac{p}{2}}}.$$

These values being substituted in the foregoing equation, we get

$$\mathrm{M}^2 + \beta^2 x^2 \mathrm{N}^2 = (1+x^2)^p \,.\, \mathrm{R}^2:$$

And if R be transformed into a function of $x^2$, we shall obtain

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)(1 + c_2^2 x^2)^2 (1 + c_4^2 x^2)^2 \ldots (1 + c_{p-2}^2 x^2)^2,$$

the new symbol $c_{2s}$ being determined by this formula,

$$\overline{c_{2s}}^2 = 1 - k^2 \sin^2 \lambda_{2s}.$$

The last equation may be resolved into these two,

$$M + \beta x N \sqrt{-1} = (1 + x\sqrt{-1})(1 + c_2 x\sqrt{-1})^2 \ldots (1 + c_{p-2} x\sqrt{-1})^2,$$

$$M - \beta x N \sqrt{-1} = (1 - x\sqrt{-1})(1 - c_2 x\sqrt{-1})^2 \ldots (1 - c_{p-2} x\sqrt{-1})^2;$$

the second of which being divided by the first, there will result,

$$\frac{1 - \tan\psi\sqrt{-1}}{1 + \tan\psi\sqrt{-1}} = \frac{1 - x\sqrt{-1}}{1 + x\sqrt{-1}} \cdot \left(\frac{1 - c_2 x\sqrt{-1}}{1 + c_2 x\sqrt{-1}}\right)^2 \ldots \left(\frac{1 - c_{p-2} x\sqrt{-1}}{1 + c_{p-2} x\sqrt{-1}}\right)^2.$$

Now $u$ being an arc of a circle, we have this well known formula,

$$u = \frac{1}{2\sqrt{-1}} \times \log. \left(\frac{1 - \tan u\sqrt{-1}}{1 + \tan u\sqrt{-1}}\right):$$

wherefore, if we take the logarithms of the factors of the foregoing expression, and substitute the equivalent circular arcs, we shall obtain,

$p$ being an odd number,

$$\psi = \varphi + 2\varphi_2 + 2\varphi_4 + 2\varphi_6 \ldots 2\varphi_{p-1}, \qquad (10)$$

the arc $\varphi_{2s}$ being determined by the equation,

$$\tan \varphi_{2s} = c_{2s} \times \tan \varphi.$$

When $p$ is an even number, we have this equation in § 5,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And, using N and M to denote the products of the binomials in the numerator and denominator of the equation (8), we have

$$\tan \psi = \frac{\beta \tan \phi \, N}{M}.$$

By the substitution of $\frac{x}{\sqrt{1 + x^2}}$ for z as before, it will be found that,

$$\beta z \sqrt{1 - z^2} \; . \; P = \frac{\beta x N}{(1 + x^2)^{\frac{p}{2}}}, \quad Q = \frac{M}{(1 + x^2)^{\frac{p}{2}}}.$$

Wherefore we have,

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)^p (1 - k^2 x^2) R^2;$$

and by converting $(1 - k^2 x^2) \cdot R^2$ into a function of $x^2$, we get

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)(1 + k'^2 x^2)(1 + c_2^2 x^2)^2 \ldots (1 + c_{p-2}^2 x^2)^2,$$

$$k'^2 = 1 - k^2, \quad \overline{c_{2n}}^2 = 1 - k^2 \sin^2 \lambda_{2n}.$$

By treating this equation as before, we get

$$\frac{1 - \tan\psi \sqrt{-1}}{1 + \tan\psi \sqrt{-1}} = \frac{1 - x\sqrt{-1}}{1 + x\sqrt{-1}} \cdot \frac{1 - k'x\sqrt{-1}}{1 + k'x\sqrt{-1}} \left(\frac{1 - c_2 x\sqrt{-1}}{1 + c_2 x\sqrt{-1}}\right)^2 \ldots;$$

and from this we deduce,

$p$ being an even number,

$$\psi = \varphi + \varphi' + \varphi_2 + \varphi_4 \ldots + \varphi_{p-2}, \qquad (11)$$

$$\tan\varphi' = k' \tan\varphi, \quad \tan\varphi_{2n} = c_{2n} \tan\varphi.$$

8. In what goes before, our attention has been confined to two related functions, which, for the sake of abridging, we have denoted by the prefixes H and K; but as we shall have occasion, in what follows, to compare several functions differing from one another in their moduli and amplitudes, it will be proper to adopt the usual and more general notation, by means of the characteristic F prefixed to the modulus and amplitude. According to this notation, the equation (1) will be thus written,

$$F(h, \psi) = \beta F(k, \varphi); \qquad F(k, \varphi) = \frac{1}{\beta} F(h, \psi).$$

The modulus $k$ being given, we can compute the amplitudes, $\lambda_1$, $\lambda_2$, &c., at least by approximation; and the amplitude $\varphi$ being supposed known, the foregoing formulas will determine the modulus $h$, the multiplier $\beta$, and the amplitude $\psi$; so that the function $F(k, \varphi)$ will be reduced to the similar function $F(h, \psi)$, of which the modulus $h$ is less than the given modulus $k$. And in like manner as the three quantities $h$, $\beta$, $\psi$ were determined from the two $k$, $\varphi$, we can deduce, from the two $h$, $\psi$, three new quantities, $h_,$, $\beta_,$, $\psi_,$ which will satisfy the equations,

$$F(h_,, \psi_,) = \beta_, F(h, \psi); \quad F(k, \varphi) = \frac{1}{\beta\beta_,} \times F(h_,, \psi_,);$$

the modulus $h_1$ being less than the modulus $h$. Continuing the like operations, we can pass along a scale of decreasing moduli, till we arrive at one which, being as small as we please, will make the function $\mathrm{F}(k, \varphi)$ approach to a circular arc as near as may be required.

If we wish to apply the same theorem to reduce the given function $\mathrm{F}(h, \varphi)$ to a logarithm, through a scale of increasing moduli, the process is not so direct. For, in the first place, the greater modulus $k$ is not immediately deducible from the less $h$, by means of the formulas that have been investigated; and, in the second place, the amplitude $\varphi$ cannot be found when $\psi$ is given without solving an equation of $p$ dimensions. The theorem is, no doubt, mathematically sufficient for effecting the reduction; but the operations required are practically impossible, except in a few cases when $p$ is a small number. But the ingenuity of M. Jacobi has provided a remedy for this inconvenience by a new transformation, which we shall now briefly explain, as it discloses a new set of remarkable properties of the elliptic functions.

If we put $y = \tan\psi$, $x = \tan\varphi$, $h'^2 = 1 - h^2$, $k'^2 = 1 - k^2$, the differential of the equation (1) will assume this form,

$$\frac{dy}{\sqrt{1 + y^2 \cdot 1 + h'^2 y^2}} = \frac{\beta\, dx}{\sqrt{1 + x^2 \cdot 1 + k'^2 x^2}};$$

and, for solving this equation, we shall have by the formula (7),

$p$ being an odd number,

$$y = \beta x \times \frac{1 - \frac{x^2}{\tan^2 \lambda_2}}{1 - \frac{x^2}{\tan^2 \lambda_1}} \cdot \frac{1 - \frac{x^2}{\tan^2 \lambda_4}}{1 - \frac{x^2}{\tan^2 \lambda_3}} \cdots \frac{1 - \frac{x^2}{\tan^2 \lambda_{p-1}}}{1 - \frac{x^2}{\tan^2 \lambda_{p-2}}}.$$

But if this value of $y$ solve the differential equation, it will still solve it, if we change $+x^2$ and $+y^2$ into $-x^2$ and $-y^2$; for it is obvious that, if the expression of $y$ make the two sides of the equation identical in one case, it will necessarily make them identical in the other case. Wherefore the equation

$$\frac{dy}{\sqrt{1 - y^2 \cdot 1 - h'^2 y^2}} = \frac{\beta\, dx}{\sqrt{1 - x^2 \cdot 1 - k'^2 x^2}}$$

will have for its solution.

$$y = \beta x \times \frac{1 + \frac{x^2}{\tan^2 \lambda_2}}{1 + \frac{x^2}{\tan^2 \lambda_1}} \cdot \frac{1 + \frac{x^2}{\tan^2 \lambda_4}}{1 + \frac{x^2}{\tan^2 \lambda_3}} \cdots \frac{1 + \frac{x^2}{\tan^2 \lambda_{p-1}}}{1 + \frac{x^2}{\tan^2 \lambda_{p-1}}}.$$

In this equation the values of $y$ and $x$ are between 0 and $\pm 1$, which limits they both attain at the same time. If we make $x = \pm 1$, and attend to the value of $\beta$, we shall find $y = \pm 1$. Let $y = \sin \tau$, $x = \sin \sigma$: then the integral of the differential equation will be

$$\left.\begin{aligned} &\mathrm{F}(k', \tau) = \beta\, \mathrm{F}(k', \sigma) \\ &\sin \tau = \beta \sin \sigma \times \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_2}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_1}} \cdot \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_4}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_3}} \cdots \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_{p-1}}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_{p-2}}}; \end{aligned}\right\} \quad (12)$$

the amplitudes $\tau$ and $\sigma$ increasing together from zero, and becoming equal to one another at 90°, and at every multiple of 90°.

A property of considerable importance in this theory, results from the comparison of the equations (1) and (12). Recalling the notations before used, viz. $\mathrm{K} = \mathrm{F}\left(k, \frac{\pi}{2}\right)$ and $\mathrm{H} = \mathrm{F}\left(h, \frac{\pi}{2}\right)$, we obtain from what bas already been said in § 1,

$$p \times \mathrm{H} = \beta \times \mathrm{K}:$$

and if we put similarly $\mathrm{K}' = \mathrm{F}\left(k', \frac{\pi}{2}\right)$ and $\mathrm{H}' = \mathrm{F}\left(h', \frac{\pi}{2}\right)$, and observe that in the equations (12), $\tau$ and $\sigma$ are equal to 90° at the same time, we shall have,

$$\mathrm{H}' = \beta\, \mathrm{K}'.$$

By combining the two equations, we readily obtain, first,

$$\frac{\mathrm{H}}{\mathrm{H}'} = \frac{1}{p} \cdot \frac{\mathrm{K}}{\mathrm{K}'}; \quad \beta = p \cdot \frac{\mathrm{H}}{\mathrm{K}}; \quad (13)$$

and secondly,

$$\beta\beta' = p; \quad \frac{\mathrm{K}'}{\mathrm{K}} = \frac{1}{p} \cdot \frac{\mathrm{H}'}{\mathrm{H}}; \quad \beta' = p \cdot \frac{\mathrm{K}'}{\mathrm{H}'}. \quad (14)$$

For any number $p$, the first of the formulas (13) determines $h$, and the second determines $\beta$, when $k$ is given. Both the formulas involve transcendent quantities; they are nevertheless of great practical utility in this theory; and they

express succinctly the conditions necessary, in order that the transformations in the equations (1) and (12) take place. A little attention will show that the formulas (14) and (13) are entirely similar, the quantities $\beta'$, $h'$, $k'$ occupying the same places in the first, that $\beta$, $k$, $h$ do in the other. From this we learn that the equations (1) and (12) will still be true if we change $\beta$, $k$, $h$ for $\beta'$, $h'$, $k'$, respectively. Thus we have,

$$\mathrm{F}(k', \psi) = \beta' \mathrm{F}(h', \varphi), \tag{15}$$

the letters $\psi$ and $\varphi$, it need hardly be noticed, although used on a former occasion, here express simply the variable amplitudes of the related functions. If therefore we divide $\mathrm{H}' = \mathrm{F}\left(h', \frac{\pi}{2}\right)$ into $p$ equal parts, and put $\mu_1$, $\mu_2$, $\mu_3$, &c., for the respective amplitudes of $\frac{1}{p}\mathrm{H}'$, $\frac{2}{p}\mathrm{H}'$, $\frac{3}{p}\mathrm{H}'$, &c.; we shall have by the formulas (4) and (9),

$$\left.\begin{aligned} \beta' &= \frac{\sin\mu_2 \sin\mu_4 \ldots \sin\mu_{2p-2}}{\sin\mu_1 \sin\mu_3 \ldots \sin\mu_{2p-1}}, \\ k' &= h'^p \cdot \left(\sin\mu_1 \sin\mu_3 \ldots \sin\mu_{2p-1}\right)^2. \end{aligned}\right\} \tag{16}$$

The multipliers $\beta$ and $\beta'$ being similar functions, the first of the amplitudes $\lambda_1$, $\lambda_2$, $\lambda_3$, &c., and the other of the amplitudes $\mu_1$, $\mu_2$, $\mu_3$, &c., the equation $\beta\beta' = p$, expresses a curious property of those functions.

And, in like manner, if we change $\beta$, $k$, $h$, respectively for $\beta'$, $h'$, $k'$ in the equation (12); or, which is the same thing, if we derive an equation from (15) in the same manner that (12) was obtained from (1), we shall get

$$\left.\begin{aligned} &\mathrm{F}(k, \tau) = \beta' \mathrm{F}(h, \sigma), \\ \sin\tau = \beta' \sin\sigma \times &\frac{1 + \frac{\sin^2\sigma}{\tan^2\mu_2}}{1 + \frac{\sin^2\sigma}{\tan^2\mu_1}} \cdot \frac{1 + \frac{\sin^2\sigma}{\tan^2\mu_4}}{1 + \frac{\sin^2\sigma}{\tan^2\mu_3}} \cdots \frac{1 + \frac{\sin^2\sigma}{\tan^2\mu_{p-1}}}{1 + \frac{\sin^2\sigma}{\tan^2\mu_{p-2}}}. \end{aligned}\right\} \tag{17}$$

Although, in the investigations of this §, we have supposed that $p$ is an odd number, yet it is obvious that they will succeed equally when $p$ is an even number, the formula (8) being used instead of (7).

The analysis by which the equation (12), of which those that follow are consequences, has been deduced from the equation (1), is precisely that by which

the expression of a circular arc is made to pass into a logarithm; so that the whole of this analytical theory rests on one principle, namely, the analogy which an elliptic function bears to a circular arc and to a logarithm, which are its extreme limits.

10. Of the transformations in the last §, the principal one is contained in the formulas (17), which constitute what is called the second theorem of M. Jacobi. One of its chief uses is to supply the defect of the first theorem by furnishing a direct process for reducing an elliptic function to a logarithm, through a scale of increasing moduli. In the function $\mathrm{F}(h, \sigma)$, the modulus $h$ being given, we know $h'$ $(= \sqrt{1 - h^2})$ named the complement of $h$ for the sake of abridging; we shall therefore obtain the amplitudes $\mu_1$, $\mu_2$, &c., by the subdivision of the function $\mathrm{H}' = \mathrm{F}\left(h', \frac{\pi}{2}\right)$; we next compute the quantities $\beta'$ and $k'$ by the formulas (16); and, the amplitude $\sigma$ being given, we deduce from the formulas (17), the amplitude $\tau$, which will satisfy the equation,

$$\mathrm{F}(h, \sigma) = \frac{1}{\beta'} . \mathrm{F}(k, \tau),$$

the modulus $k$ of the new function being greater than $h$, because the complement $k'$ is less than the complement $h'$. Taking now $k'$ the complement of $k$, we deduce from it, by means of the formulas (16) and (17), the three quantities $\beta_1'$, $k_1'$, $\tau_1$, in like manner as $\beta'$, $k'$, $\tau$ were deduced from $h'$; and we shall have these equations,

$$\mathrm{F}(k, \tau) = \beta_1' \mathrm{F}(k_1, \tau_1); \quad \mathrm{F}(h, \sigma) = \frac{1}{\beta' \beta_1'} \times \mathrm{F}(k_1, \tau_1);$$

the modulus $k_1$ being greater than $k$, because the complement $k_1'$ is less than $k'$. The like operations being continued, we shall at length arrive at a modulus $k_n$, as near the limit 1 as may be required.

Another use of the second theorem, when combined with the first, is to find any multiple of an elliptic function, or any aliquot part of it. By the first theorem, we have

$$\mathrm{F}(h, \psi) = \beta \mathrm{F}(k, \varphi);$$

and by the second, making $\sigma = \psi$ in the equations (17),

$$\mathrm{F}(k, \tau) = \beta' \mathrm{F}(h, \psi);$$

and, by combining the two equations, observing that $\beta\beta' = p$, we get

$$\mathrm{F}(k, \tau) = p\,\mathrm{F}(i, \varphi).$$

If $p$ be an odd number, the amplitudes are obtained by the formulas (2) and (17), viz.

$$\sin\psi = \beta \sin\varphi \times \frac{1 - \frac{\sin^2\varphi}{\sin^2\lambda_2}}{1 - k^2\sin^2\varphi\sin^2\lambda_2} \cdot \cdots \cdot \frac{1 - \frac{\sin^2\varphi}{\sin^2\lambda_{p-1}}}{1 - k^2\sin^2\varphi\sin^2\lambda_{p-1}},$$

$$\sin\tau = \beta' \sin\psi \times \frac{1 + \frac{\sin^2\psi}{\tan^2\mu_2}}{1 + \frac{\sin^2\psi}{\tan^2\mu_1}} \cdot \cdots \cdot \frac{1 + \frac{\sin^2\psi}{\tan^2\mu_{p-1}}}{1 + \frac{\sin^2\psi}{\tan^2\mu_{p-2}}}.$$

When a multiple is required, we pass directly, by means of the two equations, from the given amplitude $\varphi$ to $\tau$ which is sought. In the case of an aliquot part, the amplitude $\tau$ being given, the solution of the second equation, of which $p$ is the dimensions, will determine $\sin\psi$; and the amplitude $\varphi$ which is sought, will then be found by solving the first equation, which is also of $p$ dimensions. From the nature of the second equation, it has only one real root, and $p - 1$ impossible roots, for every real value of $\sin\tau$; and therefore it follows from the first equation, that the amplitude $\varphi$ of the function $\frac{1}{p}\mathrm{F}(k, \tau)$ admits in all of $p^2$ values, of which only $p$ values are real quantities, and the rest impossible.

If $p$ be an even number, the expression of $\sin\psi$ will contain radical quantities, but instead of it we may take the value of $\tan\psi$ in the formula (8); and the two equations for the amplitudes will be,

$$\tan\psi = \frac{\beta\tan\varphi}{1 - \frac{\tan^2\varphi}{\tan^2\lambda_1}} \cdot \frac{1 - \frac{\tan^2\varphi}{\tan^2\lambda_2}}{1 - \frac{\tan^2\varphi}{\tan^2\lambda_3}} \cdot \cdots \cdot \frac{1 - \frac{\tan^2\varphi}{\tan^2\lambda_{p-2}}}{1 - \frac{\tan^2\varphi}{\tan^2\lambda_{p-1}}},$$

$$\sin\tau = \frac{\beta'\sin\psi}{1 + \frac{\sin^2\psi}{\tan^2\mu_1}} \cdot \frac{1 + \frac{\sin^2\psi}{\tan^2\mu_2}}{1 + \frac{\sin^2\psi}{\tan^2\mu_3}} \cdot \cdots \cdot \frac{1 + \frac{\sin^2\psi}{\tan^2\mu_{p-2}}}{1 + \frac{\sin^2\psi}{\tan^2\mu_{p-1}}};$$

from which the same general properties may be deduced, as when $p$ is an odd number.

11. We have now demonstrated, as was proposed, the principal and leading points of this theory, for which we are indebted to M. JACOBI. For the subordinate details, and for many curious and important collateral researches that have been suggested by the new views laid open in this branch of analysis, we must refer to M. JACOBI's own work, to the papers of M. ABEL, and to the writings of LEGENDRE. We shall conclude this paper by applying the formulas that have been investigated to two particular instances, taking for $p$ the most simple values, namely 2 and 3.

Example 1. Supposing $p = 2$.

By the formulas (3) and (6) we have these equations between the amplitudes $\psi$ and $\varphi$, $z$ being $= \sin \varphi$,

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}}; \quad \cos \psi = \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{\sqrt{1 - k^2 z^2}};$$

wherefore

$$1 - k^2 z^2 = \beta^2 z^2 (1 - z^2) + \left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2;$$

and

$$\beta^2 = \frac{1}{\sin^4 \lambda_1}; \quad \frac{2}{\sin^2 \lambda_1} - \beta^2 = k^2.$$

We now get

$$k^2 = 1 - k^2; \quad \beta = 1 + k; \quad \sin^2 \lambda_1 = \frac{1}{1 + k}.$$

Also, by the formula (9),

$$h = k^2 \sin^4 \lambda_1 = \frac{k^2}{(1 + k)^2} = \frac{1 - k}{1 + k};$$

from which we deduce

$$(1 + h)(1 + k) = 2.$$

The equation (1), viz.

$$\mathrm{F}(h, \psi) = \beta \, \mathrm{F}(k, \varphi),$$

may now be put in one or other of these two forms,

$$\mathrm{F}(k, \varphi) = \frac{1 + h}{2} \mathrm{F}(h, \psi),$$

$$\mathrm{F}(h, \psi) = (1 + k) \, \mathrm{F}(k, \varphi):$$

by the first we pass from the greater modulus $k$ to the less $h$; and by the second, from the less modulus $h$ to the greater $k$.

In the first of the two cases we must derive the amplitude $\psi$ from $\varphi$: and, for this purpose we immediately obtain from the formula (11),

$$\tan(\psi - \varphi) = k' \tan \varphi.$$

Wherefore, if

$$k, h, h_1, h_2, \&c.$$

represent a series of decreasing moduli, of which the complements are,

$$k', h', h_1', h_2', \&c.$$

the successive quantities being derived from one another by these formulas,

$$h = \frac{1-k'}{1+k'}, \quad h_1 = \frac{1-h'}{1+h'}, \quad h_2 = \frac{1-h_1'}{1+h_1'}, \&c.:$$

and, if we likewise deduce a series of amplitudes in this manner,

$$\tan(\psi - \varphi) = k' \tan \varphi$$
$$\tan(\psi_1 - \psi) = h' \tan \psi$$
$$\tan(\psi_2 - \psi_1) = h_1' \tan . \psi_1, \&c.$$

we shall have these successive transformations, by which the value of the given function $\mathrm{F}(k, \varphi)$ is made to approach indefinitely to the arc of a circle,

$$\mathrm{F}(k, \varphi) = \frac{1+h}{2} \mathrm{F}(h, \psi)$$

$$\mathrm{F}(k, \varphi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} \mathrm{F}(h_1, \psi_1)$$

$$\mathrm{F}(k, \varphi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} \cdot \frac{1+h_2}{2} \cdot \mathrm{F}(h_2, \psi_2), \&c.$$

In the second case, when we would pass from the less modulus $h$ to the greater $k$, the amplitude $\varphi$ must be deduced from $\psi$. For this purpose we have

$$\frac{\sin \psi}{\cos \psi} = \frac{\beta z \sqrt{1 - z^2}}{1 - \frac{z^2}{\sin^2 \lambda_1}} = \frac{\beta \sin \varphi \cos \varphi}{1 - \beta \sin^2 \varphi};$$

but $\beta = \frac{2}{1+h}$; $\cos\varphi \sin\varphi = \frac{\sin 2\varphi}{2}$; and $1 - 2\sin^2\varphi = \cos 2\varphi$: wherefore,

$$\frac{\sin\psi}{\cos\psi} = \frac{\sin 2\varphi}{h + \cos 2\varphi}; \text{ and, } \sin(2\varphi - \psi) = h \sin\psi.$$

Wherefore if the quantities,

$$h, k, k_1, k_2, \&c.$$

represent a series of increasing moduli derived from one another by these equations,

$$k' = \frac{1-h}{1+h}, \qquad k_1' = \frac{1-k}{1+k}, \qquad k_2' = \frac{1-k_1}{1+k_1}, \&c.;$$

and further, if the amplitudes $\psi, \varphi, \varphi_1, \varphi_2$, &c. be deduced from the formulas,

$$\sin(2\varphi - \psi) = h \sin\psi,$$
$$\sin(2\varphi_1 - \varphi) = k \sin\varphi,$$
$$\sin(2\varphi_2 - \varphi_1) = k_1 \sin\varphi_1 : \&c.$$

we shall have these transformations in which the successive moduli tend to the limit 1,

$$\mathrm{F}(h, \psi) = (1 + k')\,\mathrm{F}(k, \varphi),$$
$$\mathrm{F}(h, \psi) = (1 + k')(1 + k_1')\,\mathrm{F}(k_1, \varphi_1),$$
$$\mathrm{F}(h, \psi) = (1 + k')(1 + k_1')(1 + k_2')\,\mathrm{F}(k_2, \varphi_2), \&c.$$

Example 2. Supposing $p = 3$.

By the formulas (2) and (5) we have these equations between the amplitudes $\psi$ and $\varphi$,

$$\sin\psi = \frac{\beta z\left(1 - \frac{z^2}{\sin^2\lambda_2}\right)}{1 - k^2 z^2 \sin^2\lambda_2}; \quad \cos\psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_1}}{1 - k^2 z^2 \sin^2\lambda_2}:$$

wherefore,

$$(1 - k^2 z^2 \sin^2\lambda_2)^2 = \beta^2 z^2 \left(1 - \frac{z^2}{\sin^2\lambda_2}\right)^2 + (1 - z^2)\left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2:$$

from which we get

$$2k^2 \sin^2\lambda_2 = \frac{2}{\sin^2\lambda_1} + 1 - \beta^2,$$

$$k^4 \sin^4 \lambda_2 = \frac{1}{\sin^4 \lambda_1} + \frac{2}{\sin^2 \lambda_1} - \frac{2\beta^2}{\sin^2 \lambda_2},$$

$$\beta = \frac{\sin^2 \lambda_2}{\sin^2 \lambda_1}.$$

Now, observing that $\frac{\beta^2}{\sin^2 \lambda_2} = \frac{\beta}{\sin^2 \lambda_1}$, we obtain by equating the values of $k^4 \sin^4 \lambda_2$,

$$\left(\frac{1}{\sin^2 \lambda_1} + \frac{1-\beta^2}{2}\right)^2 = \frac{1}{\sin^4 \lambda_1} + \frac{2(1-\beta)}{\sin^2 \lambda_1};$$

from which we deduce

$$\beta = \frac{2}{\sin \lambda_1} - 1.$$

In order to simplify the formulas I shall put $\frac{1}{\sin \lambda_1} = 1 + \varepsilon$: then $\beta = 1 + 2\varepsilon$, and $\sin^2 \lambda_2 = \beta \sin^2 \lambda_1 = \frac{1+2\varepsilon}{(1+\varepsilon)^2}$: and, having substituted these values in the first of the foregoing equations, we shall get,

$$\varepsilon^4 + 2\varepsilon^3 - 2k'^2\varepsilon - k'^2 = 0.$$

This equation may be resolved by the usual method into the two following quadratic factors,

$$\xi^3 = 4k^2k'^2,$$

$$\varepsilon^2 + (1 + \sqrt{1-\xi})\varepsilon - \tfrac{1}{2}\sqrt{\xi^2 + 4k'^2} - \tfrac{1}{2}\xi = 0,$$

$$\varepsilon^2 + (1 - \sqrt{1-\xi})\varepsilon + \tfrac{1}{2}\sqrt{\xi^2 + 4k'^2} - \tfrac{1}{2}\xi = 0.$$

The second of these equations has two impossible roots: the first has two real roots, one being negative and foreign to the question, and the other positive, which solves the problem. Thus $\varepsilon$ has only one value, which may be constructed geometrically, but the algebraic expression of it need not be written down.

We have next to derive the amplitude $\psi$ from $\phi$. We readily obtain from the foregoing biquadratic equation,

$$k^2 = \frac{(1+\varepsilon)^2(1-\varepsilon)}{1+2\varepsilon};$$

and hence,

$$1 - k^2 \sin^2 \lambda_2 = c_2^2 = \varepsilon^2.$$

And the formula (10) will determine $\psi$ when $\varphi$ is given.

Finally, therefore, we have these determinations,

$$\beta = 1 + 2\varepsilon, h = \frac{k^2}{(1+\varepsilon)^4}, \tan\left(\frac{\psi - \varphi}{2}\right) = \varepsilon \tan \varphi;$$

$$\mathrm{F}(k, \varphi) = \frac{1}{1+2\varepsilon} \times \mathrm{F}(h, \psi),$$

the modulus $h$ being less than $k^3$. By repeating the like operations, a succession of moduli rapidly decreasing may be formed, by means of which the given elliptic function will be reduced to a circular arc as near as may be required.

## XXI. *On the Tides in the Port of London.* *By* J. W. Lubbock, *Esq.*, *V.P. and Treas. R.S.*

Read June 16, 1831.

I HAVE the honour to present to the Society a discussion of observations of the tides made at the London Docks, in the form of various Tables, which show the time and height of high water, not only at different points of the moon's age, but also for the different months of the year, for every minute of the moon's parallax, and for every three degrees of her declination. This work has been accomplished by Mr. Dessiou of the Admiralty; but for the arrangement of the Tables, and the methods employed, I alone am responsible.

The tides take place in the river Thames with extreme regularity, and, as the rise is considerable*, the observations are made with facility. Those upon which the annexed Tables are founded are made at the entrance to the London Docks; the time of high water, or the time when the water has just made its mark, is there noted on a slate by the watchman on the pier-head, generally only to the nearest five minutes; this is afterwards copied in a book kept for the purpose by Mr. Peirse. I am enabled, through the kindness of Mr. Solly, the worthy Chairman of the London Dock Company, to present to the Society the books containing the observations which serve as the foundation of Mr. Dessiou's Tables. These observations are not made with sufficient care; but they are valuable from the extent of time during which they have been carried on, as they were instituted soon after the opening of the Docks in 1804, and have been continued without interruption to the present time. I am not aware of any series of observations of the tides so extensive, except that made at Brest by order of the French Government. Mr. Peirse informs me that the observations of the night are generally more correct than those of

* I believe about nineteen feet. I am not however able to speak with precision, not having yet been able to examine the observations of low water.

the day, (which I was not aware of until lately,) because the persons in attendance at night have nothing else to do than to look out for high water; but in the day, owing to the press of business at the top of the tide, the water has fallen an inch or two at times before it has been noticed. The water remains stationary for some little time at high water, and therefore it is difficult to fix the time of that phenomenon precisely, without the aid of some mechanical contrivance; this will however I hope soon be supplied both at the London and St. Katherine Docks.

I have also to present to the Society some observations of the tides made during one year at the East India Docks, under the superintendence of Captain Eastfield, and which were kindly undertaken at my suggestion: these observations were made with great care, and may, I believe, generally be depended upon to the minute.

When the variations in the time and height of high water due to changes in the parallax and declination of the luminaries are neglected, the theories of Bernoulli and of Laplace lead to the same results, and these results agree most remarkably with observation. The same agreement is manifested in the circumstance that the variations of the interval between the time of the moon's transit and the time of high water, and the variations of the height of high water from the mean height, are greatest at the equinoxes, and least at the solstices, or soon after; and also that the height of high water increases as the distance of the moon decreases, that is, as her parallax increases. The difference between the heights when her parallax is greatest and least may be considered nearly a foot at the London Docks.

The changes in the declinations of the sun and moon, and in the parallax of the moon, have a sensible effect both on the time and on the height of high water; the law however of these changes is so complicated, and the observations are so imperfect, that for the present I think it will be necessary to have recourse to empirical tables for their calculation. They are not in conformity with the theory of Bernoulli, which has, I believe, never before been compared with observation; nor can I reconcile them to the theory of Laplace. The constants which might be supposed the same for this port and for Brest, determined by means of these observations, do not agree with those determined by Laplace from the observations at Brest. The comparison would be ex-

tremely interesting, if the results of the observations at Brest were arranged in Tables similar to those which accompany this paper. In a classification of this kind it is necessary that the epoch of the phenomena should be defined by the minute of the moon's transit; the day of the moon's age is quite insufficient. A complete classification of this kind is desirable, because erroneous conclusions may be drawn from isolated portions of the curve; and it is necessary, in order to ascertain the agreement between theory and the fact, to compare them throughout the whole period or extent of the inequality whose effect is considered.

According to the theory of Bernoulli, and the Tables founded upon it, which are given in various works on navigation *, the variation in the interval between the time of the moon's transit and the time of high water, due to changes in the moon's parallax, vanishes or is equal to zero when the moon passes the meridian at 2 o'clock or 8 o'clock (or when $\theta - \theta_1 - \lambda + \lambda_1 = 0$ or 180°). This is directly contrary to our observations, according to which the variation is the greatest at this age of the moon. See Table XV.

According to our observations, the time and the height of high water are the same, whether the moon's declination be north or south; if any difference obtain, it must be determined by more delicate means: nor is there any sensible difference whether the moon's transit be superior or inferior, as may be seen by Table XIII., where the observations for one month are separated, those corresponding to the superior and those corresponding to the inferior transits being kept distinct.

The Tables XV, XVI, XVII, XVIII, XIX and XX are intended to furnish the means of calculating empirically the time and height of high water, taking into account the changes in the different months of the year, and also those due to changes in the moon's parallax and declination. By means of these Tables, Mr. Dessiou has calculated the times and heights of high water for the year 1826; and, having classed all the observations made at the London Docks in that year according to the different winds, has found for each a correction, which is given in Table XXI. The variation or correction due to the wind depends, of course, also on its rapidity or force; it appears however

* I have had occasion in the Preface to my "Account of the Traité sur le Flux et Réflux de la Mer," of Daniel Bernoulli, to point out an error which exists in several Tables of this kind, from the heading being reversed; but this error does not remedy the discrepancy in question.

generally to have but little effect on the tide. North-westerly gales raise the tide; south-westerly winds depress it.

Two maps are annexed to this paper, drawn by Mr. Walker; the one showing the time of high water on the coast of Great Britain, the other throughout the world, or at least in as many places as it has been ascertained.

The following are the authorities from which these maps have been compiled.

The Chart of the British Isles is taken from the observations and surveys of Messrs. M'Kenzie, Spence and Murray on the South coast of England; from those of Captain Hewett and Mr. Thomas on the East coast; France and the North Sea have been copied from the Admiralty Charts; the coast of Scotland is from M'Kenzie, Captain Huddart and the Admiralty Charts; and Ireland has been drawn from the surveys of Captain White, Captain Huddart, Mr. Nimmo, &c. &c.

In the Chart of the World, the Coast of France, Spain and Portugal is copied from the Admiralty Charts and the surveys of Tofino; Africa from those of Captain Owen; Newfoundland is from Captain Bullock; Nova Scotia and the coast of the United States are from the Admiralty Charts; South America and the west coast of North America have been copied from the Spanish Charts, and from the surveys of Captains King, Beechey, Vancouver, &c. &c. The coasts of Persia, India and the Indian Seas have been taken chiefly from Captain Horsburgh's Charts and Book of Directions; Australia is from the surveys of Captain Flinders, King, &c. &c.

It will be seen that the continents alter the direction of the *cotidal* lines, (I mean the series of points at which it is high water at the same instant,) and that the progress of the tide is not always from east to west: in the Atlantic it is from south to north; so that it is high water at nearly the same time on the coast of Portugal and on the opposite coast of America. This remarkable circumstance is noticed by Bacon*.

The map of the world is offered as a mere sketch, for our information on this subject is at present lamentably deficient. The map of Great Britain presents

* De Fluxu et Refluxu Maris. Bacon's Works, vol. ii. p. 81. "Idque non fortuito notatum, sed de industria inquisitum atque repertum, aquas ad littora adversa Europæ et Floridæ iisdem horis ab utroque littore refluere, neque deserere littus Europæ cum advolvantur ad littora Floridæ, more aquæ (ut supra diximus) agitatæ in pelvi, sed plane simul ad utrumque littus attolli et demitti."

more details, and is, I trust, more accurate; the time of high water being generally well ascertained (as a first approximation) and easily observed on these coasts. Lines are drawn round the coast to mark the depth of the water: it is to be regretted that this method of indicating the soundings, which, according to Captain Alexander *, is adopted by the Russians in their charts, is not more general. The tide which reaches the port of London is principally due to the tide which descends along the eastern coast of Scotland and England: this branch of the tide meets the tide which comes up the Channel off the sand called the Kentish Knock.

If when the tide is single, the height of the water is represented by a series of cosines, affected with constant coefficients; when the tide results from the union of two branches, each of this kind, the height of the water will still be represented by a series of cosines of angles, affected with other constant coefficients, the periods of the inequalities being the same as before, but the epochs, or the times when they arrive at their maxima and minima, different. This is shown by Laplace in the Méc. Cel. vol. ii. p. 225. Fresnel has applied the same method (Mémoire sur la Diffraction de la Lumière, p. 279.) to finding the resultant of any number of luminous waves of the same length. This method rests upon the superposition or coexistence of small oscillations, which may not be rigorously true; but it may be stated generally, that when the tide results from the union of any number of partial tides, the resultant of the whole may be considered as one tide, affected with inequalities, of which the periods are the same as the periods of the inequalities of the partial tides of which it is composed, but of which the magnitudes and the epochs are different †. It is owing to this cause that the high water in some places takes place only once in twenty-four hours, which is a case of interference.

I shall now proceed to explain the manner in which the annexed Tables were formed.

Table I. was made by classifying all the transits of the moon which took place in the years 1808 to 1826 inclusive, and in each half hour; thus, all the transits were found from the Nautical Almanac which took place between twelve o'clock (A.M.) and half-past twelve (A.M.), the moon being above the

* See Alexander's Travels to the Seat of War in the East, vol. ii. p. 101.

† If $A\cos(\theta - \lambda)$ represent any inequality, the constant $A$ determines its magnitude, the variable $\theta$ its period, and the constant $\lambda$ its epoch.

horizon; the inferior transits of the moon between the same limits were interpolated, and added to the rest. The mean of these transits for the month of January, with the equation of time, was found to be twenty-four minutes, and the mean corresponding time of high water $2^h\ 9^m$; the transits between half-past twelve and one, and in every succeeding half-hour, were treated in the same way. The columns underneath are given to explain more fully the method employed, which was the same for all the Tables.

All the details of these calculations are deposited in the library of the Society.

TABLE I.

| | Time of Moon's Transit from Nautical Almanac. | | | | Time of corresponding High Water from the Dock Books. | | | |
|---|---|---|---|---|---|---|---|---|
| | January. 0 to 30′. | | January. 30′ to 1 hour. | | | | | |
| | January. | h. m. | January. | h. m. | January. | h. m. | January. | h. m. |
| 1808 | 28 | 0 10 int$^d$. | 28 | 0 33 | 28 | 2 15 | 28 | 2 40 |
| —— | .... | ..... | 29 | 0 56 int$^d$. | .... | .... | 29 | 2 45 |
| 1809 | 16 | 0 21 | 17 | 0 48 int$^d$. | 16 | 2 0 | 17 | 2 30 |
| 1810 | 6 | 0 19 int$^t$. | 6 | 0 49 | 6 | 2 0 | 6 | 2 30 |
| 1811 | 25 | 0 11 int$^d$. | 25 | 0 41 | 25 | 2 0 | 25 | 2 30 |
| 1812 | 14 | 0 6 | 15 | 0 35 int$^t$. | 14 | 2 0 | 15 | 2 35 |
| 1813 | 3 | 0 13 int$^d$. | 3 | 0 39 | 3 | 2 30 | 3 | 2 25 |
| 1814 | 22 | 0 20 int$^t$. | 22 | 0 44 | 22 | 2 30 | 22 | 2 45 |
| 1815 | 11 | 0 20 int$^d$. | 11 | 0 44 | 11 | 2 15 | 11 | 2 40 |
| 1816 | 29 | 0 10 | 29 | 0 34 int$^d$. | 29 | 2 35 | 29 | 2 45 |
| —— | .... | ..... | 30 | 0 57 | .... | .... | 30 | 3 0 |
| 1817 | 17 | 0 2 | .... | ..... | 17 | 1 45 | .... | .... |
| —— | 18 | 0 30 int$^t$. | 18 | 0 56 | 18 | 2 15 | 18 | 2 25 |
| 1818 | 7 | 0 5 int$^d$. | 7 | 0 37 | 7 | 1 45 | 7 | 2 15 |
| 1819 | 26 | 0 2 int$^d$. | 26 | 0 33 | 26 | 1 45 | 26 | 2 0 |
| 1820 | 16 | 0 23 int$^d$. | 16 | 0 55 | 16 | 2 15 | 16 | 2 30 |
| 1821 | 4 | 0 17 | 5 | 0 47 int$^d$. | 4 | 2 10 | 5 | 2 20 |
| 1822 | 23 | 0 16 | 24 | 0 41 int$^d$. | 23 | 2 20 | 24 | 2 35 |
| 1823 | 12 | 0 7 | 13 | 0 31 int$^d$. | 12 | 2 15 | 13 | 2 30 |
| —— | .... | ..... | 13 | 0 54 | .... | .... | 13 | 2 45 |
| 1824 | 1 | 0 8 | 2 | 0 32 int$^t$. | 1 | 2 10 | 2 | 2 25 |
| —— | 31 | 0 12 | 2 | 0 55 | 31 | 2 35 | 2 | 2 30 |
| 1825 | 19 | 0 14 | 20 | 0 36 int$^t$. | 19 | 1 50 | 20 | 2 30 |
| —— | .... | ..... | 20 | 0 58 | .... | .... | 20 | 3 0 |
| 1826 | 8 | 0 4 | 9 | 0 56 | 8 | 2 0 | 9 | 2 40 |
| —— | 9 | 0 30 int$^d$. | .... | ..... | 9 | 2 20 | .... | .... |
| | 22) | 300 | 24) | 1071 | 22) | 47 30 | 24) | 61 30 |
| | | 13$\frac{3}{4}$ | | 44$\frac{1}{2}$ | Mean corresponding Time of High Water. | 2 9 | ..... | 2 34 |
| Equation of Time*.... | | 10 | ..... | 10 | | | | |
| Mean time of transit... | | 24 | ..... | 54$\frac{1}{2}$ | | | | |

* The equation of time was generally taken for the 15th of the month.

Hence I infer that neglecting the influence of the moon's parallax and declination, when the moon passes the meridian at twenty-four minutes past twelve mean solar time in January, the time of high water at the London Docks is nine minutes past two.

The other Tables, which result immediately from the observations, were formed in the same manner, and are, I trust, sufficiently explained by the heading which accompanies each.

| Let $m$ be the mass | In the notation of the Méc. Cel. |
|---|---|
| $\delta$ . . declination | $L$ |
| $\theta$ . . hour angle | $v$ |
| $r$ . . distance from the centre of the earth | $nt + \varpi - \psi$ |
| $\Pi$ . . mean horizontal parallax } of the sun. | $r$ |
| $l$ . . longitude | $\varphi$ |
| $n$ . . mean motion in its orbit | $m$ |
| $\omega$ . . obliquity of the ecliptic | $\varepsilon$ |

The same letters accented at foot refer to the moon, $\omega$, being the inclination of her orbit to the equator; let also $\phi$ denote the geographical latitude of the port, in the notation of the Méc. Cel. $90° - \theta$.

Considering only the terms which are multiplied by the cube of the parallax, the forces which produce the phenomena of the tides are the partial differences of the function

$$\frac{3m}{2r^3}\left\{(\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\theta)^2 - \frac{1}{3}\right\}$$ See the Méc. Cel. vol. v. p. 168.

According to the theory of Laplace, this function being equal to $\Sigma A\cos(\theta - \lambda)$, $\theta$ being any variable angle depending on the time, and $A$ and $\lambda$ constants, the height of the water at any given time is equal to $\Sigma(A'\cos(\theta - \lambda'))$, $\theta$ being the same angle as before, and $A'$ and $\lambda'$ other constants.

$$\left\{\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\theta\right\}^2 = \frac{\sin^2\varphi}{2} + \frac{\cos^2\varphi}{4} + \frac{\cos^2\varphi}{4}\cos 2\theta - \left\{\frac{\sin^2\varphi}{2} - \frac{\cos^2\varphi}{4}\right\}\cos 2\delta$$

$$+ \frac{\cos^2\varphi}{8}\left\{\cos(2\theta - 2\delta) + \cos(2\theta + 2\delta)\right\} + \frac{\sin 2\varphi}{2}\sin 2\delta\cos\theta.$$

If the longitude of the sun be introduced by putting for $\delta$ its value from the equation

$$\sin \delta = \sin \omega \sin l$$

$l$ being reckoned from the first point of Aries,

$$\left\{\sin\varphi \sin\delta + \cos\varphi\cos\delta\cos\theta\right\}^2 = \frac{\cos^2\varphi}{2}\left(1-\frac{\sin^2\omega}{2}\right) + \frac{\sin^2\varphi\sin^2\omega}{2}$$
$$+ \frac{\cos^2\varphi}{2}\left(1-\frac{\sin^2\omega}{2}\right)\cos 2\theta$$
$$+ \frac{\cos^2\varphi\sin^2\omega}{8}\left\{\cos(2\theta - 2l) + \cos(2\theta + 2l)\right\}$$
$$+ \sin^2\omega\left\{\frac{\cos^2\varphi}{4} - \frac{\sin^2\varphi}{2}\right\}\cos 2l$$
$$+ \frac{\sin 2\varphi}{2}\sin 2\delta\cos\theta$$

Considering the results of many years so as to destroy the effects of changes in the moon's parallax and declination, and taking the mean of the times of high water when the moon passes the meridian at any given time, and twelve hours later, the tides, owing to the united action of the sun and moon, depend on the terms

$$\frac{3m\Pi^3\cos^2\varphi}{4}\left(1-\frac{\sin^2\omega}{2}\right)\cos 2\theta$$
$$+ \frac{3m\Pi^3\cos^2\varphi\sin^2\omega}{16}\left\{\cos(2\theta - 2l) + \cos(2\theta + 2l)\right\}$$
$$+ \frac{3m\Pi^3\sin^2\omega}{2}\left\{\frac{\cos^2\varphi}{4} - \frac{\sin^2\varphi}{2}\right\}\cos 2l$$
$$+ \frac{3m_1\Pi_1^3\cos^2\varphi}{4}\left(1-\frac{\sin^2\omega_1}{2}\right)\cos 2\theta_1$$

and the height of the water will be represented by

$$A m \Pi^3\left(1-\frac{\sin^2\omega}{2}\right)\cos(2\theta - 2\lambda)$$
$$+ \frac{Bm\Pi^3}{4}\sin^2\omega\left\{\cos(2\theta - 2l - 2\lambda) + \cos(2\theta + 2l - 2\lambda)\right.$$
$$+ C\Pi^3\sin^2\omega\cos 2l$$

$$+A_i m_i \Pi_i^3 \left(1-\frac{\sin^2\omega_i}{2}\right)\cos(2\theta_i-2\lambda_i)$$

This expression coincides with that given by Laplace, (Méc. Cel. vol. v. p. 169, at foot,) when the terms multiplied by $\sin^4\frac{\omega}{2}$ are neglected, $nt+\varpi-mt$ (in the notation of the Méc. Cel.) being equal to the hour angle $\theta$ at the equinoxes and solstices, that is, when $l=0$, 90°, 180° or 270°; but Laplace neglects the term $\cos 2l$.

Laplace supposes that

$$A=(1+nx)B \qquad A_i=(1+n_ix)B$$

$n$ being the mean motion of the luminary in its orbit, and $x$ an indeterminate quantity, to be determined by the observations. Making these substitutions, differentiating the expression for the height to find the time of high water, and supposing $\mathrm{d}\theta=\mathrm{d}\theta_i$ and $\omega=\omega_i$,

$$\tan(2\theta_i-2\lambda_i)=\frac{\dfrac{m\Pi^3}{m_i\Pi_i^3}\dfrac{(1+nx)}{(1+n_ix)}\sin(2\theta_i-2\theta-2\lambda_i+2\lambda)+\dfrac{m\Pi_3}{2m_i\Pi_i^3}\dfrac{\sin^2\omega\cos 2l}{(1+n_ix)\left(1-\frac{\sin^2\omega}{2}\right)}\sin(2\theta_i-2\theta-2\lambda_i+2\lambda)+C'\sin(2\theta_i-2\lambda_i-2l)}{1+\dfrac{m\Pi^3}{m_i\Pi_i^3}\dfrac{(1+nx)}{(1+n_ix)}\cos(2\theta_i-2\theta-2\lambda_i+2\lambda)+\dfrac{m\Pi^3}{2m_i\Pi_i^3}\dfrac{\sin^2\omega\cos 2l}{(1+n_ix)\left(1-\frac{\sin^2\omega}{2}\right)}\cos(2\theta_i-2\theta-2\lambda_i+2l)+C'\cos(2\theta_i-2\lambda_i-2l)}$$

$C'$ being a constant different from $C$, $\theta$ and $\theta_i$ being the values of those variables at the instant of high water.

If we consider the mean of all the months of the year, column A, Table III.

$$\tan(2\theta_i-2\lambda_i)=\frac{\dfrac{m\Pi^3}{m_i\Pi_i^3}\dfrac{(1+nx)}{(1+n_ix)}\sin(2\theta_i-2\theta-2\lambda_i+2\lambda)}{1+\dfrac{m\Pi^3}{m_i\Pi_i^3}\dfrac{(1+nx)}{(1+n_ix)}\cos(2\theta_i-2\theta-2\lambda_i+2\lambda)}$$

The constants which enter into this expression may be determined by means of column A. The mean of this column is $1^h\ 25^m$; I therefore take

$$\lambda_i=1^h\ 25^m, \qquad \lambda-\lambda_i=2^h, \qquad \lambda=3^h\ 25^m$$

$\dfrac{m\Pi^3(1+nx)}{m_i\Pi_i^3(1+n_ix)}$ = tangent of twice the difference of the interval when the moon passes the meridian at $2^h$ and at $5^h$. I take

$$\log.\frac{m\Pi^3(1+nx)}{m_i\Pi_i^3(1+n_ix)}=9{\cdot}5784858$$

When the moon passes the meridian at three o'clock, $\theta - \theta_1 - \lambda + \lambda_1 = 15°$.

| | |
|---|---|
| log. sin 30° = 9.6989700 | log. cos 30° = 9.9375806 |
| 9.5784858 | 9.5784858 |
| 9.2774558 | 9.5160164 = log. ·3281 |
| log. 1.3281 = .1232308 | |
| 9.1542250 = log. tan 8° 7′ or 32′ 38″ in time | |

$2\theta_1 - 2\lambda_1 = -32'\ 38''$, $\theta_1 - \lambda_1 = 16\frac{1}{4}'$, $\theta_1 = 1^h\ 25^m - 16^m = 1^h\ 9^m$

In this way the following Table was calculated.

| Time of Moon's Transit. | Interval between the Moon's Transit and the Time of High Water. | | Error of Calculation. |
|---|---|---|---|
| | Observed. | Calculated. | |
| h | h. m. | h. m. | m. |
| 0 | 1 57 | 1 56 | — 1 |
| 1 | 1 42 | 1 41 | — 1 |
| 2 | 1 26 | 1 25 | — 1 |
| 3 | 1 11 | 1 9 | — 2 |
| 4 | 56 | 54 | — 2 |
| 5 | 45 | 44.1 | — 0.9 |
| 6 | 42 | 41 | — 0 |
| 7 | 52 | 53.7 | + 1.7 |
| 8 | 1 23 | 1 25 | + 2 |
| 9 | 1 56 | 1 56.3 | + 0.3 |
| 10 | 2 10 | 2 9 | — 1 |
| 11 | 2 8 | 2 6.5 | 1·5 |

According to Table III. the establishment * of the London Docks is $1^h\ 57^m$. Adding ten minutes, the establishment of London Bridge is $2^h\ 7^m$. The establishment of the London Docks according to Mr. Bulpit, who has calculated the times and heights of high water in the river for many years, is $2^h$. Mr. Bulpit's calculations are founded on tables constructed by the late Captain Huddart, which have not been published.

In the Philosophical Transactions, vol. xiii. p. 10, Flamsteed gives "A correct Tide-table, showing the true times of the high waters at London Bridge to every day in the year 1683." It appears from his remarks that the earliest Tide-tables which were calculated for the Port of London were made on the supposition that the tide always followed three hours after the moon's transit.

* By the establishment of any port is meant the time of high water at new and full moon.

The Annuaire du Bureau des Longitudes and various works on navigation give $2^h\ 45^m$ for the establishment of the Port of London; it seems therefore probable that the high water takes place now much earlier than it did formerly. It is generally admitted that the constant $\lambda - \lambda_i$ is the same at different ports; this, however, requires to be confirmed by accurate determinations, and is one of the most interesting questions in the theory of the tides. BERNOULLI makes this constant 20° or $1^h\ 20^m$ only in time; LAPLACE adopts the same value, though not expressly. BERNOULLI's Table for finding the time of high water, which is given in the Annuaire du Bureau des Longitudes, and in works on navigation, is quite inapplicable to the Port of London on this account, even in the mean distances of the moon, as the following comparison will show:

| Time of Moon's Transit. | Interval between the Moon's Transit and the Time of High Water. | | Error of Calculation. |
|---|---|---|---|
| | Observed. | Calculated. | |
| h | h m | h m | m |
| 0 | 1 57 | 1 57 | 0 |
| 2 | 1 26 | 1 23½ | − 2½ |
| 4 | 56 | 55 | − 1 |
| 6 | 42 | 54½ | + 12½ |
| 8 | 1 23 | 2 0 | + 37 |
| 10 | 2 10 | 2 20 | + 10 |

The Table in the Annuaire for the year 1829 was made use of; the Table in that for 1831 differs from that only in form.

If $\lambda = \lambda_i$ the greatest tide takes place at new and full moon.

The quantity $\lambda$ is called by LAPLACE the fundamental hour of the port.—Exposition du Système du Monde, p. 289.

I shall now compare the heights of high water, calculated by means of the same constants with those given by observation column B, Table V.

According to the preceding theory, the height of high water

$$= D + E\{\cos 2(\theta_i - \lambda_i) + .3788 \cos 2(\theta - \lambda)\}$$

$D$ and $E$ being constants to be determined by observation.

When the moon passes the meridian at two o'clock, $2\theta_i - 2\lambda_i = 0$, $2\theta - 2\lambda = 0$

——————— eight o'clock, $2\theta_i - 2\lambda_i = 0$, $2\theta - 2\lambda = 180°$

Hence by column B. Table V.

$$22.80 = D + 1.3788\,E$$
$$19.43 = D + .6212\,E$$

Whence $D = 16.68$ and $\log. E = .64819$, $E = 4{\cdot}448$

When the moon passes the meridian at

| h. | h. | | | |
|---|---|---|---|---|
| 2 | | the height of high water | $= 16.68 + 4{\cdot}448\{1 + .3788\} = 22.80$ | |
| 3 or | 1 | do. | do. | $= 16.68 + 4.448\{\cos 8°\ 7' + .3788 \cos 22°\} = 22.64$ |
| 4 or | 0 | do. | do. | $= 16.68 + 4.448\{\cos 15°\ 25' + .3788 \cos 44°\ 30'\} = 22.17$ |
| 5 or | 11 | do. | do. | $= 16.68 + 4.448\{\cos 20°\ 45' + .3788 \cos 69°\ 30'\} = 21.42$ |
| 6 or | 10 | do. | do. | $= 16.68 + 4.448\{\cos 22°\ 2' - .3788 \sin 8°\} = 20.56$ |
| 7 or | 9 | do. | do. | $= 16.68 + 4.448\{\cos 15°\ 45' - .3788 \sin 44°\} = 19.79$ |
| 8 | | do. | do. | $= 16.68 + 4.448\{1 - .3788\} = 19.43$ |

The following Table gives a comparison between theory and observation.

| Time of Moon's Transit. | Height of High Water. | | Error of Calculation. |
|---|---|---|---|
| | Observed. | Calculated. | |
| h. | Ft. | Ft. | Ft. |
| 0 | 22.46 | 22.17 | − .29 |
| 1 | 22.72 | 22.64 | − .08 |
| 2 | 22.80 | 22.80 | 0 |
| 3 | 22.59 | 22.64 | + .05 |
| 4 | 22.10 | 22.17 | + .07 |
| 5 | 21.28 | 21.42 | + .14 |
| 6 | 20.37 | 20.56 | + .19 |
| 7 | 19.56 | 19.79 | + .23 |
| 8 | 19.43 | 19.43 | 0 |
| 9 | 20.10 | 19.79 | − .31 |
| 10 | 20.92 | 20.56 | − .36 |
| 11 | 21.85 | 21.42 | − .43 |

I shall now endeavour to show that the expression for $\tan(2\theta_1 - 2\lambda_1)$, p. 387, line 13, does not satisfy the observations.

When the moon passes the meridian at 2 o'clock $\theta - \theta_1 - \lambda + \lambda_1 = 0$.

$$\tan(2\theta - 2\lambda_1) = \frac{-C' \sin 2I}{1 + C' \cos 2I} \text{ nearly.}$$

When the moon passes the meridian at 8 o'clock, $\theta - \theta_1 - \lambda + \lambda_1 = 90°$

$$\tan(2\theta_1 - 2\lambda_1) = \frac{-C' \sin 2I}{1 - \frac{m\,\Pi^3}{m_1 \Pi_1^3} + C' \cos 2I} \text{ nearly.}$$

At the end of May $l = 45°$ nearly, and then according to Table III. when the moon passes the meridian at 2, the interval between her transit and H. W. is $1^h\ 16^m$ nearly; when the moon passes the meridian at 8, the interval between her transit and H. W. is $1^h\ 38^m$ nearly. The variation or difference being in the one case $-5^m$, ($1^h\ 20^m - 1^h\ 25^m = -5^m$), and in the other $+13^m$, ($1^h\ 38^m - 1^h\ 25^m = +13^m$).

The formula gives in the one case

$$\tan(2\theta_1 - 2l_1) = -C'$$

and in the other

$$\tan(2\theta_1 - 2l_1) = \frac{-C'}{1 - \frac{m\Pi^3}{m_1\Pi_1^3}}$$

according to which the variation or difference should be in both cases of the same sign: hence the variations or differences $-5^m$ and $+13^m$ are not owing to this inequality.

It is possible that this discrepancy may be owing to the terms which have $r_1^4$ in the denominator; but the observations are made so carelessly that I shall not attempt to take these terms into consideration.

The discrepancy which I have pointed out is sufficient to show that Table III. cannot be represented by the expression of p. 387, line 13, whatever value be given to the constants employed: this discrepancy is not confined to the epoch I have noticed, but extends throughout the year.

Neglecting this discrepancy which not only prevents the expression above from representing the observations, but must also I think vitiate any conclusions to be drawn respecting the mass of the moon; in April when $l = 0$, and

when the moon passes the meridian at $5^h$, the interval is $0^h\ 34^m$

. . . . . 11 . . . . . 2 14

$$2^h\ 14^m - 34^m = 1^h\ 40^m$$

I may suppose

$$\tan 1^h\ 40^m = \tan 25° = \frac{m\Pi^3(1+nx)}{m_1\Pi_1^3(1+n_1x)} + \frac{2m_1\Pi_1^3}{m\Pi^3}\frac{\sin^2\omega}{(1+n_1x)\left(1-\frac{\sin^2\omega}{2}\right)}$$

$$\cdot 4663 = \cdot 3788 + \frac{m\Pi^3}{2m_1\Pi_1^3}\frac{\sin^2\omega}{(1+n_1x)\left(1-\frac{\sin^2\omega}{2}\right)}$$

from which equation combined with the equation

$$\frac{m\,\Pi^3}{m_{,}\Pi_{,}^3}\,\frac{(1+nx)}{(1+n_{,}x)} = .3788$$

$1 + nx$ may be determined.

The theory of LAPLACE differs essentially from that of BERNOULLI, in supposing the constants to be modified by local circumstances, which consideration introduces the factor $\frac{1+nx}{1+n_{,}x}$.

According to the observations at Brest, $1+nx = 1.01891, 1+n_{,}x = 1.25291$

$$\log.\frac{m\,\Pi^3}{m_{,}\Pi_{,}^3}\,\frac{(1+nx)}{(1+n_{,}x)} = 9.5385031\,? \qquad \frac{m\,\Pi^3}{m_{,}\Pi_{,}^3}\,\frac{(1+nx)}{(1+n_{,}x)} = .3455.$$

Whence

$$\log.\frac{m\,\Pi^3}{m_{,}\Pi_{,}^3} = 9.6283227$$

which gives the mass of the moon equal to that of the earth divided by 74.946.

If $m_{,,}$ be the mass of the earth, by an extension of the third law of KEPLER, $\frac{(m+m_{,,})\,\Pi^3}{(m_{,}+m_{,,})\,\Pi_{,}^3}$ is nearly equal to the ratio of the squares of the periodic times of the moon about the earth, and of the earth about the sun, or of their mean motions. This ratio is known very accurately to be .0748013. Hence neglecting $m_{,,}$ with regard to $m$, we have

$$\frac{m\,\Pi^3}{(m_{,}+m_{,,})\,\Pi_{,}^3} = (.0748013)^2.$$

If we neglect the factor $\frac{1+nx}{1+n_{,}x}$ which is nearly unity;

$$\log.\frac{m\,\Pi^3}{m_{,}\Pi_{,}^3} = 9.57846$$

$$\log.\frac{m\,\Pi^3}{(m_{,}+m_{,,})\,\Pi_{,}^3} = 7.74782 \qquad \log.\frac{m_{,}+m_{,,}}{m_{,}} = 1.83064$$

$$\frac{m_{,}}{m_{,}+m_{,,}} = \frac{1}{67.7},$$

which gives the mass of the moon equal to that of the earth, divided by 66.7.

Astronomers will, no doubt, concur in following the opinion of LAPLACE. "En considérant la pétitesse des quantités qui m'ont servi à déterminer l'ac-

croissement de l'action lunaire et en réflechissant que ces quantités sont du même ordre que les pétites erreurs dont l'application du principe de la coéxistence des ondulations très pétites aux phénomènes des marées est susceptible; je n'ose garantir l'exactitude de cette valeur de la masse lunaire, et j'incline à penser que les phénomènes astronomiques sont plus propres à le fixer."

A very slight change in the interval employed produces a considerable alteration in the mass of the moon.

After differentiation $\mathrm{d}\theta$ was supposed $= \mathrm{d}\theta_1$, but since the time of the moon's synodic revolution is 29.530 days, and that in this time the hour angle of the moon is less by one circumference than that of the sun,

$$\mathrm{d}\theta = \frac{29.530}{28.530}\,\mathrm{d}\theta_1$$

The observations of the times and height of high water in different months of the year may be nearly represented by neglecting in the expression for the height, the term of which the argument is $2\theta + 2l - 2\lambda$, but it is not easy to see why the coefficient of this inequality differs so much from that of the inequality of which the argument is $2\theta - 2l - 2\lambda$, the cause of this difference must be clearly established before the theory can be considered complete.

If $l_1$ be the longitude of the moon reckoned from her perigee, and increased by a constant, considering the terms depending on the changes of the moon's parallax, the height of the water may be represented by

$$m\Pi^3\cos(2\theta - 2\lambda) + m_1\Pi_1^3\left\{\cos(2\theta_1 - 2\lambda_1) + C\left(\cos(2\theta_1 - 2\lambda_1 - l_1) + \cos(2\theta_1 - 2\lambda_1 + l_1)\right)\right\}$$

or

$$m\Pi^3\cos(2\theta - 2\lambda) + m_1\Pi_1^3\{1 + C\cos l_1\}\cos(2\theta_1 - 2\lambda_1)$$

$$\tan(2\theta_1 - 2\lambda_1) = \frac{\dfrac{m\Pi^3}{m_1\Pi_1^3}\,\dfrac{\sin(2\theta_1 - 2\theta - 2\lambda_1 + 2\lambda)}{(1 + C\cos l_1)}}{1 - \dfrac{m\Pi^3}{m_1\Pi_1^3}\,\dfrac{\cos(2\theta_1 - 2\theta - 2\lambda_1 + 2\lambda)}{(1 + C\cos l_1)}}$$

According to this formula, the variation in the interval which elapses between the time of the moon's transit, and the term of high water, is equal to zero when the moon passes the meridian at 2 o'clock or 8 o'clock; and this is the case whatever value be given to the constant $C$, or the constant which I have sup-

posed to be included in the angle $l_1$: this result is directly contrary to observation, as may be seen by Table VIII. This difficulty may be got over by supposing the coefficients $(C)$ of the angles $2\theta_1 - 2\lambda_1 - l_1$ and $2\theta_1 - 2\lambda_1 + l_1$ to be different; it remains however to assign the cause of this difference.

Not having been able to satisfy myself with respect to this and other discrepancies, I shall not attempt to form any Tables from theory for the purpose of calculating the effects due to changes in the declination of the luminaries, and in the parallax of the moon; but I hope that no great length of time will elapse before the results contained in M. Dessiou's Tables will be confirmed by more accurate observations here and elsewhere, and that the problem of the tides will meet with the attention it deserves, no less practically than in connection with Physical Astronomy, and particularly with the determination, even if imperfectly, of the mass of the moon. "Les marées ne sont pas moins interessantes à connoître, que les inégalités des mouvemens célestes. On a négligé pendant long-temps de les suivre avec une éxactitude convenable, à cause des irrégularités qu'elles présentent; mais ces irrégularités disparaissent en multipliant les observations."—Exp. Syst. du Monde, p. 289.

The Tables XV. XVI. &c. present irregularities, owing to the comparative paucity and the imperfection of the observations. M. Dessiou has therefore formed Tables from these, in which these irregularities are arbitrarily removed: these Tables are published in the Companion to the British Almanac for 1832, and no doubt represent the phenomena rather better than the former. As, however, these alterations are arbitrary, I have preferred giving here the results of the observations without any change. The observations employed were taken in all cases indiscriminately from the Dock books.

I have now only to express my acknowledgements to the Committee of the Society for the Diffusion of Useful Knowledge, and to Mr. Pond the late superintendent of the Nautical Almanac, who have given me M. Dessiou's assistance in forming the annexed Tables.

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XXI. Showing the influence of the direction of the Wind on the time and height of High Water.

## Table I.

Showing the time of High Water at the London Docks corresponding to the mean time
vations made between the 1st of January

| January. | | | February. | | | March. | | | April. | | | May. | | | June. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. P.M. | Corresponding H.W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H.W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H.W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H.W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H.W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H.W. | No. of Obs. |
| h m | h m | | h m | h m | | h m | h m | | h m | h m | | h m | h m | | h m | h m | |
| 0 24 | 2 9 | 22 | 0 26 | 2 17.8 | 19 | 0 23.5 | 2 25.8 | 26 | 0 17.5 | 2 17.8 | 23 | 0 12 | 2 5.5 | 23 | 0 15.8 | 2 3 | 19 |
| 0 54.5 | 2 34 | 24 | 0 58 | 2 45 | 25 | 0 53.2 | 2 48 | 26 | 0 47 | 2 38.2 | 20 | 0 43.5 | 2 26.5 | 21 | 0 44.2 | 2 25 | 20 |
| 1 24 | 3 1 | 20 | 1 27 | 3 7.5 | 25 | 1 23.5 | 3 5 | 22 | 1 15 | 2 57.2 | 23 | 1 12.5 | 2 46.8 | 21 | 1 14.2 | 2 46.5 | 21 |
| 1 55 | 3 20 | 24 | 1 59 | 3 31.5 | 21 | 1 53.5 | 3 32 | 28 | 1 46 | 3 21 | 23 | 1 42.2 | 3 11.5 | 22 | 1 48 | 3 10.5 | 23 |
| 2 24 | 3 47 | 27 | 2 28.5 | 3 51.5 | 25 | 2 24.2 | 3 52.5 | 23 | 2 16.5 | 3 42 | 22 | 2 11.2 | 3 29.5 | 22 | 2 16 | 3 32.2 | 20 |
| 2 55 | 4 8 | 22 | 3 0 | 4 18.2 | 25 | 2 55 | 4 14.2 | 25 | 2 46.8 | 4 4.5 | 20 | 2 42.2 | 3 51.5 | 21 | 2 45 | 3 54.5 | 22 |
| 3 24 | 4 33 | 27 | 3 30 | 4 43 | 24 | 3 25 | 4 32.5 | 24 | 3 14 | 4 23 | 19 | 3 10.5 | 4 9.5 | 20 | 3 14 | 4 18.2 | 22 |
| 3 54 | 4 59 | 27 | 3 59 | 5 7.2 | 20 | 3 55.5 | 4 54 | 22 | 3 43.2 | 4 39 | 19 | 3 42 | 4 35.8 | 26 | 3 45 | 4 42 | 24 |
| 4 26 | 5 26 | 24 | 4 27.8 | 5 26.5 | 22 | 4 26.2 | 5 19 | 24 | 4 13 | 5 3 | 23 | 4 11 | 4 56 | 22 | 4 15 | 5 8 | 23 |
| 4 55 | 5 50 | 25 | 5 3.2 | 5 54.2 | 25 | 4 53.8 | 5 37.2 | 20 | 4 43 | 5 20.2 | 19 | 4 42 | 5 23 | 21 | 4 46 | 5 37.2 | 26 |
| 5 25 | 6 25 | 26 | 5 31.5 | 6 18.5 | 22 | 5 23.8 | 6 0.8 | 20 | 5 11.5 | 5 45.2 | 21 | 5 11 | 5 44.8 | 25 | 5 17 | 6 8 | 24 |
| 5 56 | 6 54 | 28 | 6 0.2 | 6 39.8 | 19 | 5 56 | 6 23.2 | 24 | 5 43.8 | 6 11 | 22 | 5 43 | 6 19.5 | 24 | 5 46 | 6 39 | 23 |
| 6 26 | 7 19 | 23 | 6 29 | 7 15 | 21 | 6 24 | 6 50.8 | 21 | 6 15.8 | 6 43 | 25 | 6 9.8 | 6 45.5 | 20 | 6 16 | 7 14 | 25 |
| 6 55 | 7 57 | 24 | 7 0.5 | 7 44.2 | 19 | 6 55 | 7 24.3 | 23 | 6 47 | 7 17.8 | 20 | 6 39 | 7 22.5 | 26 | 6 46 | 7 52 | 25 |
| 7 24 | 8 30 | 24 | 7 29 | 8 22 | 19 | 7 25.5 | 7 55.2 | 21 | 7 16.5 | 8 4.5 | 24 | 7 11 | 8 16 | 28 | 7 15 | 8 34 | 22 |
| 7 56 | 9 13 | 25 | 7 58 | 8 57 | 18 | 7 55 | 8 48.2 | 23 | 7 49 | 9 2.2 | 23 | 7 41.5 | 9 8.8 | 24 | 7 46 | 9 19 | 27 |
| 8 26 | 10 1 | 21 | 8 28.2 | 9 44.5 | 22 | 8 24.5 | 9 42 | 23 | 8 15.5 | 9 49.8 | 21 | 8 12.5 | 9 54 | 26 | 8 17 | 9 58.5 | 23 |
| 8 56 | 10 41 | 23 | 8 58 | 10 34.5 | 16 | 8 55.2 | 10 37.8 | 22 | 8 41.5 | 10 34.2 | 23 | 8 43.8 | 10 48.5 | 27 | 8 46 | 10 42 | 23 |
| 9 27 | 11 22 | 20 | 9 28.5 | 11 20.2 | 23 | 9 24.2 | 11 21.5 | 22 | 9 14.8 | 11 27 | 26 | 9 12.5 | 11 22.5 | 21 | 9 16 | 11 17.5 | 23 |
| 9 54 | 11 56 | 23 | 9 59.5 | 12 4.5 | 18 | 9 54 | 12 3.5 | 25 | 9 45.2 | 12 1.5 | 24 | 9 40.5 | 11 56.5 | 24 | 9 46 | 11 51 | 19 |
| 10 25 | 12 29 | 18 | 10 29.2 | 12 34.8 | 21 | 10 24.2 | 12 38.5 | 22 | 10 15.8 | 12 32 | 24 | 10 10.8 | 12 24.5 | 25 | 10 15 | 12 21 | 23 |
| 10 53 | 12 56 | 21 | 11 0.5 | 13 11 | 22 | 10 53.8 | 13 4.5 | 25 | 10 45 | 13 0 | 25 | 10 40.5 | 12 50.5 | 22 | 10 45 | 12 48 | 20 |
| 11 24 | 13 23 | 24 | 11 30 | 13 36.5 | 18 | 11 25.2 | 13 31.2 | 27 | 11 15.5 | 13 23.5 | 22 | 11 11 | 13 18.2 | 25 | 11 14 | 13 12 | 19 |
| 11 56 | 13 48 | 21 | 11 59 | 14 0 | 21 | 11 54.2 | 13 54 | 21 | 11 45.8 | 13 53 | 26 | 11 41.2 | 13 42.5 | 21 | 11 45 | 13 39 | 22 |
| 12 26 | 14 13 | 23 | 12 28.2 | 14 20 | 21 | 12 24 | 14 21 | 28 | 12 16.5 | 14 12.2 | 22 | 12 10 | 14 1.5 | 21 | 12 15 | 14 0.5 | 20 |
| 12 56 | 14 35 | 21 | 12 58.8 | 14 42.8 | 23 | 12 54.2 | 14 38.2 | 22 | 12 46.5 | 14 34.2 | 22 | 12 40.2 | 14 26.8 | 23 | 12 45 | 14 22 | 20 |
| 13 26 | 14 58 | 22 | 13 28.5 | 15 6.2 | 23 | 13 24 | 15 6 | 25 | 13 14.8 | 14 56 | 22 | 13 11.2 | 14 46.5 | 22 | 13 14 | 14 45 | 20 |
| 13 54 | 15 19 | 21 | 13 59 | 15 26.5 | 22 | 13 55 | 15 29.5 | 27 | 13 44.5 | 15 14.5 | 21 | 13 41.5 | 15 9 | 20 | 13 45.5 | 15 9 | 23 |
| 14 26 | 15 47 | 28 | 14 29.5 | 15 52.2 | 27 | 14 25 | 15 50.5 | 20 | 14 14.5 | 15 37 | 21 | 14 10 | 15 25 | 21 | 14 17 | 15 32 | 22 |
| 14 57 | 16 11 | 23 | 15 0.2 | 16 16.5 | 20 | 14 54 | 16 10.5 | 25 | 14 44.5 | 15 59 | 22 | 14 41 | 15 46.2 | 23 | 14 47 | 15 55 | 20 |
| 15 26 | 16 33 | 23 | 15 29.5 | 16 39 | 24 | 15 23 | 16 32 | 22 | 15 16.5 | 16 19 | 21 | 15 10.5 | 16 10 | 20 | 15 15 | 16 17 | 22 |
| 15 54 | 17 3 | 25 | 16 1 | 17 1.2 | 25 | 15 52 | 16 52 | 21 | 15 47 | 16 38 | 21 | 15 38.2 | 16 30.2 | 20 | 15 46 | 16 41 | 25 |
| 16 26 | 17 31 | 27 | 16 31.5 | 17 27.8 | 20 | 16 23 | 17 16 | 26 | 16 15.5 | 17 2.2 | 19 | 16 9.2 | 16 54.8 | 25 | 16 16 | 17 10 | 22 |
| 16 55 | 17 57 | 24 | 16 59 | 17 45 | 20 | 16 55.5 | 17 38.5 | 22 | 16 45 | 17 22 | 22 | 16 44 | 17 23 | 26 | 16 45 | 17 36 | 24 |
| 17 26 | 18 25 | 27 | 17 32 | 18 12 | 23 | 17 24 | 17 57.2 | 20 | 17 16.2 | 17 45 | 22 | 17 13.2 | 17 50 | 22 | 17 15.5 | 18 6 | 26 |
| 17 56 | 18 55 | 25 | 17 59 | 18 39.2 | 19 | 17 50.5 | 18 23.8 | 20 | 17 46 | 18 13.5 | 20 | 17 40.5 | 18 16.8 | 23 | 17 46 | 18 41 | 25 |
| 18 24 | 19 31 | 24 | 18 26.5 | 18 58 | 18 | 18 23 | 18 53.8 | 24 | 18 13.2 | 18 41 | 19 | 18 10.5 | 18 50.5 | 24 | 18 15 | 19 13 | 22 |
| 18 56 | 20 2 | 24 | 18 58.5 | 19 36.8 | 25 | 18 54 | 19 22.5 | 20 | 18 44 | 19 16 | 25 | 18 40 | 19 34 | 26 | 18 45 | 19 50 | 30 |
| 19 27 | 20 40 | 23 | 19 29.5 | 20 8.5 | 17 | 19 23 | 20 1.5 | 23 | 19 15.5 | 20 5 | 24 | 19 10.5 | 20 15.5 | 24 | 19 17.5 | 20 38 | 24 |
| 19 56 | 21 22 | 23 | 19 59 | 20 50.8 | 20 | 19 53.5 | 20 46 | 22 | 19 45 | 20 54.5 | 21 | 19 40 | 21 5 | 26 | 19 46 | 21 26 | 21 |
| 20 26 | 21 57 | 22 | 20 28 | 21 39.2 | 19 | 20 23.5 | 21 44 | 33 | 20 14.5 | 21 39 | 24 | 20 11.5 | 21 58.5 | 27 | 20 14 | 21 57 | 24 |
| 20 55 | 22 44 | 20 | 20 59.2 | 22 29.2 | 20 | 20 54 | 22 40.5 | 25 | 20 45.5 | 22 37.2 | 26 | 20 41.8 | 22 46 | 24 | 20 45 | 22 42 | 25 |
| 21 24 | 23 23 | 22 | 21 33 | 23 14.2 | 19 | 21 24.3 | 23 24.8 | 22 | 21 16 | 23 22.2 | 23 | 21 11.2 | 23 23 | 26 | 21 17 | 23 23.5 | 24 |
| 21 54 | 23 56 | 21 | 21 57.5 | 23 56.2 | 19 | 21 53 | 24 4.5 | 23 | 21 45.2 | 24 3.5 | 25 | 21 42.5 | 24 0 | 25 | 21 46 | 23 55.5 | 20 |
| 22 26 | 24 39 | 23 | 22 28.8 | 24 29.5 | 21 | 22 23.5 | 24 43 | 26 | 22 13.8 | 24 29 | 22 | 22 11.2 | 24 30 | 24 | 22 15 | 24 23.5 | 22 |
| 22 57 | 25 0 | 21 | 23 0 | 25 4 | 23 | 22 55.5 | 25 7.2 | 27 | 22 45.8 | 25 4 | 30 | 22 40.2 | 24 58 | 21 | 22 47 | 24 51 | 22 |
| 23 26 | 25 26 | 21 | 23 31 | 25 31.8 | 20 | 23 26 | 25 35.5 | 23 | 23 17 | 25 31.2 | 20 | 23 8 | 25 23.5 | 26 | 23 17 | 25 17 | 21 |
| 23 55 | 25 49 | 20 | 24 0 | 25 57 | 21 | 23 54 | 25 57.6 | 22 | 23 45.8 | 25 55 | 22 | 23 41.2 | 25 44 | 22 | 23 46 | 25 40 | 20 |

## Table I.

(reckoned from noon) of the Moon's Transit in each month of the year; from 13,073 obser-
1808 and the 31st of December 1826.

| July. | | | August. | | | September. | | | October. | | | November. | | | December. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. P.M. | Corresponding H. W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H. W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H. W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H. W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H. W. | No. of Obs. | Moon's Transit. P.M. | Corresponding H. W. | No. of Obs. |
| h m | h m | | h m | h m | | h m | h m | | h m | h m | | h m | h m | | h m | h m | |
| 0 21 | 2 11 | 19 | 0 19 | 2 17 | 22 | 0 9 | 2 14 | 26 | 0 0 | 2 6 | 24 | 23 59 | 1 51 | 22 | 0 13.2 | 1 58 | 22 |
| 0 49 | 2 32 | 22 | 0 51.5 | 2 45 | 25 | 0 40 | 2 36 | 25 | 0 30 | 2 26 | 23 | 0 29 | 2 9 | 20 | 0 42 | 2 21.2 | 19 |
| 1 20 | 2 57 | 23 | 1 19 | 3 3 | 20 | 1 10.5 | 2 59 | 24 | 1 1 | 2 48 | 25 | 0 59 | 2 33 | 22 | 1 9.2 | 2 38.5 | 20 |
| 1 50 | 3 18 | 23 | 1 48 | 3 27 | 26 | 1 41 | 3 34 | 24 | 1 41 | 3 7 | 23 | 1 31 | 2 55 | 21 | 1 40.3 | 3 6 | 22 |
| 2 20 | 3 41 | 25 | 2 19 | 3 49 | 25 | 2 10 | 3 43 | 20 | 2 2 | 3 30 | 21 | 2 1 | 3 16 | 20 | 2 11 | 3 23 | 23 |
| 2 51 | 4 5 | 25 | 2 48 | 4 14 | 27 | 2 39.5 | 4 3 | 27 | 2 32 | 3 50 | 25 | 2 31 | 3 38 | 22 | 2 41 | 3 46 | 22 |
| 3 21 | 4 33 | 24 | 3 20.5 | 4 35 | 24 | 3 10 | 4 26.5 | 20 | 3 3 | 4 8 | 20 | 3 1 | 4 0 | 20 | 3 11 | 4 11 | 23 |
| 3 51 | 4 57 | 27 | 3 49.5 | 5 0 | 23 | 3 39 | 4 44 | 22 | 3 31 | 4 28 | 20 | 3 31 | 4 20 | 22 | 3 41 | 4 34 | 24 |
| 4 21 | 5 23 | 23 | 4 19 | 5 25 | 25 | 4 9 | 5 3 | 21 | 4 0 | 4 46 | 21 | 4 3 | 4 44 | 23 | 4 11.5 | 5 0 | 26 |
| 4 50 | 5 50 | 26 | 4 50 | 5 47 | 25 | 4 40 | 5 26 | 25 | 4 29 | 5 8 | 22 | 4 33 | 5 12 | 21 | 4 42 | 5 24 | 25 |
| 5 21 | 6 19 | 28 | 5 20.5 | 6 14 | 22 | 5 12.5 | 5 56 | 21 | 5 1 | 5 33 | 24 | 5 0 | 5 35.5 | 20 | 5 11.5 | 5 57 | 24 |
| 5 52 | 6 51 | 25 | 5 49 | 6 36 | 22 | 5 41.5 | 6 17 | 19 | 5 31 | 5 57.5 | 20 | 5 29 | 6 4.5 | 23 | 5 41 | 6 31 | 28 |
| 6 21 | 7 24 | 22 | 6 18 | 7 6 | 22 | 6 10.5 | 6 40 | 21 | 6 1 | 6 23 | 20 | 5 59 | 6 35 | 23 | 6 12 | 7 13 | 26 |
| 6 50 | 7 50 | 27 | 6 49 | 7 33 | 24 | 6 41 | 7 10 | 22 | 6 32 | 7 1 | 25 | 6 31 | 7 13 | 23 | 6 42 | 7 46 | 24 |
| 7 21 | 8 36 | 25 | 7 19 | 8 12 | 21 | 7 11.5 | 7 45 | 21 | 7 4 | 7 42 | 23 | 7 1 | 8 3 | 24 | 7 11 | 8 23 | 27 |
| 7 53 | 9 17 | 24 | 7 48 | 8 51.5 | 22 | 7 42 | 8 34 | 22 | 7 31.5 | 8 39 | 21 | 7 29 | 8 51 | 23 | 7 41 | 9 15 | 23 |
| 8 23 | 9 58 | 21 | 8 19 | 9 39 | 24 | 8 13 | 9 27 | 21 | 7 59.5 | 9 25 | 22 | 8 1 | 9 44 | 26 | 8 10 | 9 54 | 25 |
| 8 51 | 10 33 | 20 | 8 50 | 10 29 | 23 | 8 41 | 10 22 | 22 | 8 29.3 | 10 13 | 25 | 8 31 | 10 29 | 24 | 8 43 | 10 35 | 26 |
| 9 23 | 11 14.5 | 23 | 9 20.5 | 11 12 | 22 | 9 11.5 | 11 14 | 23 | 8 59.5 | 11 4 | 27 | 9 0 | 11 13 | 25 | 9 11 | 11 16 | 22 |
| 9 53 | 11 49 | 21 | 9 50 | 11 55.5 | 21 | 9 42 | 11 48 | 23 | 9 31 | 11 52 | 24 | 9 30 | 11 50 | 25 | 9 41 | 11 51 | 23 |
| 10 23 | 12 23 | 22 | 10 18 | 12 23.5 | 20 | 10 12 | 12 30 | 23 | 10 0 | 12 21 | 26 | 10 0 | 12 14 | 23 | 10 12 | 12 19 | 23 |
| 10 53 | 12 55 | 19 | 10 48 | 12 55.5 | 22 | 10 41.5 | 12 59 | 23 | 10 30 | 12 49 | 26 | 10 31 | 12 47 | 25 | 10 41.5 | 12 55 | 20 |
| 11 23 | 13 19 | 23 | 11 18 | 13 22 | 23 | 11 9 | 13 19 | 21 | 11 1 | 13 15 | 25 | 11 1 | 13 12 | 22 | 11 10 | 13 17 | 21 |
| 11 53 | 13 47.5 | 21 | 11 48.5 | 13 47.5 | 23 | 11 40 | 13 49 | 27 | 11 31 | 13 43 | 25 | 11 30 | 13 33 | 21 | 11 39 | 13 34 | 20 |
| 12 23 | 14 6.5 | 21 | 12 20 | 14 18 | 26 | 12 10 | 14 14 | 23 | 12 1 | 14 3 | 23 | 12 0 | 13 56 | 22 | 12 9 | 13 58 | 23 |
| 12 53 | 14 35 | 22 | 12 51 | 14 41 | 24 | 12 39 | 14 34 | 23 | 12 29.5 | 14 25 | 25 | 12 29 | 14 18 | 20 | 12 40 | 14 22 | 22 |
| 13 20 | 14 58 | 23 | 13 18.5 | 15 4 | 21 | 13 10 | 14 56 | 27 | 13 0 | 14 44 | 22 | 12 58 | 14 32 | 21 | 13 11 | 14 45 | 21 |
| 13 51 | 15 16 | 21 | 13 48 | 15 27.5 | 26 | 13 41 | 15 18.5 | 25 | 13 30 | 15 8.5 | 24 | 13 30 | 14 57 | 23 | 13 41.5 | 15 10 | 25 |
| 14 20 | 15 41.5 | 25 | 14 19 | 15 50 | 26 | 14 12 | 15 40.5 | 23 | 14 0 | 15 28 | 22 | 14 1 | 15 17 | 21 | 14 14 | 15 31 | 21 |
| 14 51 | 16 6.5 | 25 | 14 50 | 16 12 | 23 | 14 41 | 16 3 | 22 | 14 29 | 15 45 | 23 | 14 30 | 15 38 | 20 | 14 45 | 15 52 | 21 |
| 15 21 | 16 31.5 | 25 | 15 19 | 16 35 | 22 | 15 10 | 16 22 | 23 | 15 0 | 16 8 | 22 | 15 0 | 15 59.5 | 21 | 15 11 | 16 12 | 23 |
| 15 52 | 16 56.5 | 25 | 15 50 | 17 0 | 27 | 15 39.5 | 16 43 | 21 | 15 30 | 16 23 | 22 | 15 30 | 16 20 | 22 | 15 42 | 16 40 | 25 |
| 16 21 | 17 26 | 27 | 16 22 | 17 23 | 22 | 16 9 | 17 5 | 23 | 16 1 | 16 48.5 | 23 | 15 59 | 16 40 | 20 | 16 12 | 17 5 | 23 |
| 16 52 | 17 55 | 26 | 16 51.5 | 17 47.5 | 23 | 16 40 | 17 30 | 22 | 16 31 | 17 11 | 20 | 16 30 | 17 12 | 24 | 16 40.5 | 17 32 | 24 |
| 17 22 | 18 21.5 | 23 | 17 19 | 18 11.5 | 23 | 17 11 | 17 48 | 23 | 17 0.5 | 17 30 | 23 | 17 1 | 17 34 | 24 | 17 11 | 18 2 | 29 |
| 17 50 | 18 51 | 26 | 17 48 | 18 35.5 | 22 | 17 41 | 18 15 | 19 | 17 31 | 17 56 | 23 | 17 30 | 18 4 | 20 | 17 42 | 18 33 | 25 |
| 18 22 | 19 21 | 28 | 18 19.5 | 19 2 | 25 | 18 10.5 | 18 40 | 20 | 18 2 | 18 29 | 21 | 18 0 | 18 39 | 26 | 18 12 | 19 6 | 26 |
| 18 51 | 19 54 | 24 | 18 50 | 19 36 | 22 | 18 41 | 19 11 | 19 | 18 30.5 | 19 1 | 24 | 18 31.5 | 19 13 | 25 | 18 41 | 19 44 | 24 |
| 19 20 | 20 33 | 23 | 19 20 | 20 10 | 23 | 19 10.5 | 19 44 | 21 | 19 0 | 19 39 | 20 | 19 2 | 20 1.5 | 23 | 19 9.5 | 20 33 | 24 |
| 19 49 | 21 15 | 22 | 19 50 | 20 58 | 21 | 19 41.5 | 20 38 | 21 | 19 29.5 | 20 34 | 26 | 19 30 | 20 52 | 23 | 19 40 | 21 5 | 28 |
| 20 20 | 21 54 | 24 | 20 19.5 | 21 43 | 24 | 20 11 | 21 37 | 22 | 20 1 | 21 35 | 25 | 19 59.5 | 21 41 | 26 | 20 12 | 21 52 | 25 |
| 20 50 | 22 39 | 22 | 20 49 | 22 32 | 21 | 20 41 | 22 27 | 20 | 20 31 | 22 20 | 28 | 20 29 | 22 22 | 23 | 20 40 | 22 40 | 23 |
| 21 21 | 23 16.5 | 22 | 21 19 | 23 9 | 23 | 21 10 | 23 9 | 22 | 21 1 | 23 15 | 25 | 21 0.5 | 23 9 | 26 | 21 13.5 | 23 18.5 | 25 |
| 21 51 | 23 53 | 21 | 21 51 | 23 53.5 | 23 | 21 40 | 23 53 | 25 | 21 30 | 23 50 | 23 | 21 31 | 23 47 | 24 | 21 42.5 | 23 45 | 19 |
| 22 20 | 24 22 | 20 | 22 21 | 24 27 | 21 | 22 10 | 24 26 | 23 | 22 0.5 | 24 28 | 29 | 22 1.5 | 24 17 | 23 | 22 11 | 24 14 | 22 |
| 22 50 | 24 51.5 | 21 | 22 51.5 | 24 55 | 22 | 22 40 | 24 58 | 21 | 22 32 | 24 53 | 25 | 22 31 | 24 43 | 22 | 22 41.5 | 24 45 | 22 |
| 23 20 | 25 22 | 22 | 23 19 | 25 23 | 21 | 23 9.5 | 25 24 | 26 | 23 2 | 25 20 | 24 | 22 59 | 25 10 | 19 | 23 10.5 | 25 10 | 20 |
| 23 51 | 25 46 | 24 | 23 49 | 25 44 | 22 | 23 39 | 25 50 | 22 | 23 31 | 25 40 | 22 | 23 28 | 25 30 | 23 | 23 41 | 25 34 | 21 |

## Table II. (Interpolated from Table I.)

Showing the Interval between the Moon's Transit and the time of High Water at the London Docks for every month in the year.

| Moon's Transit: Mean Solar Time. | Jan. | Feb. | March. | April. | May. | June. | July. | Aug. | Sept. | Oct. | Nov. | Dec. | Mean. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. |
| P.M. | | | | | | | | | | | | | |
| h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m |
| 0 0 | 1 53 | 1 57 | 2 3 | 2 5 | 1 57 | 1 51 | 1 54 | 1 56 | 2 7 | 2 6 | 1 52 | 1 50 | 1 57 |
| 0 30 | 1 44 | 1 51 | 2 1 | 1 56 | 1 48 | 1 44 | 1 48 | 1 57 | 1 59 | 1 56 | 1 40 | 1 43 | 1 50 |
| 1 0 | 1 39 | 1 47 | 1 52 | 1 47 | 1 39 | 1 36 | 1 41 | 1 51 | 1 52 | 1 47 | 1 34 | 1 33 | 1 43 |
| 1 30 | 1 35 | 1 39 | 1 41 | 1 39 | 1 31 | 1 28 | 1 34 | 1 43 | 1 45 | 1 36 | 1 24 | 1 27 | 1 35 |
| 2 0 | 1 25 | 1 32 | 1 37 | 1 30 | 1 22 | 1 21 | 1 26 | 1 36 | 1 37 | 1 28 | 1 15 | 1 16 | 1 26 |
| 2 30 | 1 21 | 1 26 | 1 26 | 1 22 | 1 12 | 1 13 | 1 19 | 1 29 | 1 27 | 1 19 | 1 7 | 1 8 | 1 19 |
| 3 0 | 1 12 | 1 18 | 1 17 | 1 14 | 1 2 | 1 7 | 1 13 | 1 22 | 1 19 | 1 6 | 0 59 | 1 3 | 1 11 |
| 3 30 | 1 8 | 1 13 | 1 6 | 1 1 | 0 56 | 1 0 | 1 10 | 1 14 | 1 9 | 0 57 | 0 49 | 0 55 | 1 3 |
| 4 0 | 1 4 | 1 7 | 0 57 | 0 53 | 0 48 | 0 55 | 1 6 | 1 9 | 1 2 | 0 46 | 0 42 | 0 50 | 0 57 |
| 4 30 | 0 59 | 0 58 | 0 52 | 0 43 | 0 43 | 0 52 | 1 1 | 1 3 | 0 50 | 0 39 | 0 39 | 0 46 | 0 50 |
| 5 0 | 0 56 | 0 52 | 0 42 | 0 35 | 0 36 | 0 51 | 1 0 | 0 55 | 0 45 | 0 32 | 0 36 | 0 44 | 0 45 |
| 5 30 | 1 0 | 0 47 | 0 35 | 0 30 | 0 35 | 0 52 | 0 58 | 0 50 | 0 39 | 0 26 | 0 36 | 0 49 | 0 43 |
| 6 0 | 0 57 | 0 39 | 0 27 | 0 27 | 0 36 | 0 55 | 1 0 | 0 47 | 0 32 | 0 22 | 0 36 | 0 57 | 0 41 |
| 6 30 | 0 54 | 0 46 | 0 27 | 0 29 | 0 41 | 1 2 | 1 3 | 0 46 | 0 29 | 0 29 | 0 42 | 1 3 | 0 44 |
| 7 0 | 1 3 | 0 44 | 0 29 | 0 39 | 0 58 | 1 12 | 1 5 | 0 47 | 0 32 | 0 37 | 1 1 | 1 9 | 0 51 |
| 7 30 | 1 8 | 0 53 | 0 34 | 0 58 | 1 19 | 1 26 | 1 18 | 0 57 | 0 46 | 1 6 | 1 23 | 1 34 | 1 6 |
| 8 0 | 1 19 | 1 0 | 0 57 | 1 21 | 1 36 | 1 37 | 1 27 | 1 10 | 1 5 | 1 25 | 1 42 | 1 41 | 1 22 |
| 8 30 | 1 36 | 1 17 | 1 23 | 1 45 | 1 55 | 1 48 | 1 36 | 1 26 | 1 30 | 1 43 | 1 58 | 1 49 | 1 39 |
| 9 0 | 1 46 | 1 37 | 1 45 | 2 3 | 2 8 | 1 58 | 1 45 | 1 44 | 1 55 | 2 4 | 2 13 | 2 0 | 1 55 |
| 9 30 | 1 56 | 1 53 | 2 0 | 2 14 | 2 15 | 2 3 | 1 53 | 1 57 | 2 5 | 2 20 | 2 20 | 2 8 | 2 5 |
| 10 0 | 2 3 | 2 5 | 2 10 | 2 16 | 2 15 | 2 5 | 1 57 | 2 6 | 2 13 | 2 21 | 2 14 | 2 8 | 2 9 |
| 10 30 | 2 4 | 2 5 | 2 13 | 2 16 | 2 11 | 2 5 | 2 1 | 2 7 | 2 18 | 2 19 | 2 16 | 2 11 | 2 10 |
| 11 0 | 2 2 | 2 11 | 2 9 | 2 12 | 2 8 | 2 0 | 2 0 | 2 7 | 2 13 | 2 14 | 2 11 | 2 9 | 2 8 |
| 11 30 | 1 56 | 2 6 | 2 5 | 2 8 | 2 3 | 1 56 | 1 55 | 2 2 | 2 9 | 2 12 | 2 3 | 1 59 | 2 3 |
| A.M. | | | | | | | | | | | | | |
| 0 0 | 1 51 | 2 1 | 1 59 | 2 2 | 1 54 | 1 49 | 1 52 | 1 59 | 2 6 | 2 2 | 1 56 | 1 51 | 1 57 |
| 0 30 | 1 46 | 1 51 | 1 54 | 1 52 | 1 48 | 1 41 | 1 43 | 1 55 | 1 58 | 1 55 | 1 49 | 1 44 | 1 50 |
| 1 0 | 1 38 | 1 44 | 1 44 | 1 45 | 1 39 | 1 34 | 1 41 | 1 49 | 1 49 | 1 44 | 1 34 | 1 37 | 1 41 |
| 1 30 | 1 31 | 1 38 | 1 40 | 1 36 | 1 31 | 1 28 | 1 34 | 1 44 | 1 41 | 1 38 | 1 27 | 1 31 | 1 35 |
| 2 0 | 1 24 | 1 27 | 1 32 | 1 27 | 1 19 | 1 20 | 1 24 | 1 37 | 1 32 | 1 28 | 1 16 | 1 22 | 1 27 |
| 2 30 | 1 20 | 1 23 | 1 24 | 1 19 | 1 9 | 1 12 | 1 20 | 1 28 | 1 24 | 1 16 | 1 8 | 1 10 | 1 18 |
| 3 0 | 1 13 | 1 16 | 1 15 | 1 9 | 1 2 | 1 5 | 1 14 | 1 20 | 1 17 | 1 8 | 1 0 | 1 3 | 1 10 |
| 3 30 | 1 8 | 1 9 | 1 7 | 0 57 | 0 54 | 0 59 | 1 9 | 1 14 | 1 7 | 0 53 | 0 50 | 0 59 | 1 2 |
| 4 0 | 1 8 | 1 0 | 0 58 | 0 47 | 0 47 | 0 55 | 1 5 | 1 7 | 0 58 | 0 47 | 0 41 | 0 55 | 0 56 |
| 4 30 | 1 5 | 0 56 | 0 51 | 0 42 | 0 42 | 0 53 | 1 4 | 1 0 | 0 52 | 0 40 | 0 42 | 0 52 | 0 52 |
| 5 0 | 1 1 | 0 46 | 0 41 | 0 33 | 0 38 | 0 51 | 1 2 | 0 55 | 0 42 | 0 29 | 0 33 | 0 51 | 0 45 |
| 5 30 | 0 58 | 0 40 | 0 33 | 0 28 | 0 37 | 0 53 | 1 0 | 0 51 | 0 35 | 0 25 | 0 34 | 0 51 | 0 42 |
| 6 0 | 0 59 | 0 40 | 0 32 | 0 28 | 0 39 | 0 56 | 1 0 | 0 46 | 0 31 | 0 27 | 0 39 | 0 53 | 0 43 |
| 6 30 | 0 58 | 0 33 | 0 30 | 0 30 | 0 49 | 1 1 | 1 1 | 0 44 | 0 30 | 0 30 | 0 42 | 1 0 | 0 44 |
| 7 0 | 1 8 | 0 38 | 0 30 | 0 40 | 1 1 | 1 13 | 1 6 | 0 47 | 0 33 | 0 39 | 0 59 | 1 6 | 0 52 |
| 7 30 | 1 14 | 0 39 | 0 40 | 0 59 | 1 12 | 1 31 | 1 17 | 0 56 | 0 49 | 1 4 | 1 22 | 1 18 | 1 5 |
| 8 0 | 1 27 | 0 52 | 0 57 | 1 17 | 1 39 | 1 40 | 1 29 | 1 13 | 1 15 | 1 34 | 1 41 | 1 34 | 1 23 |
| 8 30 | 1 33 | 1 12 | 1 27 | 1 37 | 1 57 | 1 50 | 1 39 | 1 31 | 1 39 | 1 49 | 1 53 | 1 53 | 1 40 |
| 9 0 | 1 51 | 1 30 | 1 50 | 1 59 | 2 9 | 2 1 | 1 51 | 1 45 | 1 55 | 2 14 | 2 9 | 2 4 | 1 57 |
| 9 30 | 2 0 | 1 40 | 2 0 | 2 12 | 2 16 | 2 8 | 1 58 | 1 54 | 2 8 | 2 20 | 2 16 | 2 4 | 2 5 |
| 10 0 | 2 2 | 1 59 | 2 13 | 2 16 | 2 19 | 2 8 | 2 2 | 2 4 | 2 15 | 2 27 | 2 16 | 2 4 | 2 10 |
| 10 30 | 2 3 | 2 1 | 2 18 | 2 16 | 2 18 | 2 6 | 2 2 | 2 5 | 2 17 | 2 21 | 2 12 | 2 3 | 2 10 |
| 11 0 | 2 2 | 2 4 | 2 12 | 2 16 | 2 16 | 2 2 | 2 2 | 2 4 | 2 16 | 2 18 | 2 11 | 2 1 | 2 9 |
| 11 30 | 1 59 | 2 1 | 2 9 | 2 11 | 2 8 | 1 57 | 2 0 | 2 1 | 2 12 | 2 9 | 2 2 | 1 55 | 2 4 |

## Table III.

Formed from the preceding by taking the mean of the interval when the Moon passes the meridian at any given time, and twelve hours later.

| Moon's Transit. Mean Solar Time. | Jan. | Feb. | March. | April. | May. | June. | July. | Aug. | Sept. | Oct. | Nov. | Dec. | Mean. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | High Water. | A. |
| h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m |
| 0 0 | 1 52 | 1 59 | 2 1 | 2 4 | 1 55 | 1 50 | 1 53 | 1 57 | 2 6 | 2 4 | 1 54 | 1 50 | 1 57 |
| 0 30 | 1 45 | 1 51 | 1 57 | 1 54 | 1 48 | 1 43 | 1 45 | 1 56 | 1 58 | 1 55 | 1 44 | 1 44 | 1 50 |
| 1 0 | 1 39 | 1 45 | 1 48 | 1 46 | 1 39 | 1 35 | 1 41 | 1 50 | 1 50 | 1 45 | 1 34 | 1 35 | 1 42 |
| 1 30 | 1 33 | 1 38 | 1 40 | 1 38 | 1 31 | 1 28 | 1 34 | 1 43 | 1 43 | 1 37 | 1 26 | 1 29 | 1 35 |
| 2 0 | 1 25 | 1 29 | 1 34 | 1 29 | 1 20 | 1 20 | 1 25 | 1 37 | 1 34 | 1 28 | 1 16 | 1 19 | 1 26 |
| 2 30 | 1 20 | 1 24 | 1 25 | 1 20 | 1 10 | 1 12 | 1 19 | 1 29 | 1 25 | 1 18 | 1 8 | 1 9 | 1 18 |
| 3 0 | 1 13 | 1 17 | 1 16 | 1 12 | 1 2 | 1 6 | 1 13 | 1 21 | 1 18 | 1 7 | 1 0 | 1 3 | 1 11 |
| 3 30 | 1 8 | 1 11 | 1 7 | 0 59 | 0 55 | 1 0 | 1 9 | 1 14 | 1 8 | 0 56 | 0 50 | 0 57 | 1 3 |
| 4 0 | 1 6 | 1 3 | 0 58 | 0 50 | 0 48 | 0 55 | 1 6 | 1 8 | 1 0 | 0 47 | 0 42 | 0 53 | 0 56 |
| 4 30 | 1 2 | 0 57 | 0 52 | 0 43 | 0 42 | 0 53 | 1 3 | 1 1 | 0 51 | 0 40 | 0 40 | 0 49 | 0 51 |
| 5 0 | 0 59 | 0 49 | 0 42 | 0 34 | 0 37 | 0 51 | 1 1 | 0 55 | 0 43 | 0 31 | 0 35 | 0 47 | 0 45 |
| 5 30 | 0 59 | 0 43 | 0 34 | 0 29 | 0 36 | 0 53 | 0 59 | 0 51 | 0 37 | 0 26 | 0 35 | 0 50 | 0 43 |
| 6 0 | 0 58 | 0 40 | 0 30 | 0 28 | 0 38 | 0 56 | 1 0 | 0 47 | 0 32 | 0 24 | 0 38 | 0 55 | 0 42 |
| 6 30 | 0 56 | 0 40 | 0 29 | 0 30 | 0 45 | 1 2 | 1 2 | 0 45 | 0 30 | 0 29 | 0 42 | 1 2 | 0 44 |
| 7 0 | 1 5 | 0 41 | 0 30 | 0 40 | 0 59 | 1 13 | 1 6 | 0 47 | 0 33 | 0 38 | 1 0 | 1 8 | 0 52 |
| 7 30 | 1 11 | 0 46 | 0 37 | 0 58 | 1 16 | 1 28 | 1 18 | 0 56 | 0 47 | 1 5 | 1 22 | 1 21 | 1 5 |
| 8 0 | 1 23 | 0 56 | 0 57 | 1 19 | 1 37 | 1 38 | 1 28 | 1 12 | 1 10 | 1 29 | 1 41 | 1 37 | 1 23 |
| 8 30 | 1 35 | 1 14 | 1 25 | 1 41 | 1 56 | 1 49 | 1 38 | 1 29 | 1 34 | 1 46 | 1 55 | 1 51 | 1 39 |
| 9 0 | 1 49 | 1 33 | 1 47 | 2 1 | 2 9 | 1 59 | 1 48 | 1 45 | 1 55 | 2 9 | 2 11 | 2 2 | 1 56 |
| 9 30 | 1 58 | 1 46 | 2 1 | 2 13 | 2 16 | 2 5 | 1 55 | 1 56 | 2 6 | 2 20 | 2 18 | 2 6 | 2 5 |
| 10 0 | 2 2 | 2 2 | 2 12 | 2 16 | 2 17 | 2 7 | 1 59 | 2 5 | 2 14 | 2 24 | 2 15 | 2 6 | 2 10 |
| 10 30 | 2 3 | 2 3 | 2 15 | 2 16 | 2 14 | 2 6 | 2 2 | 2 6 | 2 17 | 2 20 | 2 14 | 2 7 | 2 10 |
| 11 0 | 2 2 | 2 7 | 2 11 | 2 14 | 2 12 | 2 1 | 2 1 | 2 6 | 2 15 | 2 16 | 2 11 | 2 5 | 2 8 |
| 11 30 | 1 57 | 2 3 | 2 7 | 2 10 | 2 5 | 1 57 | 1 58 | 2 2 | 2 10 | 2 10 | 2 3 | 1 57 | 2 3 |

## Table IV.

Showing the Height of High Water at the London Docks corresponding to the mean time of the Moon's Transit in each month of the year; from 6638 observations made between the 1st of January 1808 and the 31st of December 1826.

| January. | | | February. | | | March. | | | April. | | | May. | | | June. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. |
| h m | Feet. | | h m | Feet. | | h m | Feet. | | h m | Feet. | | h m | Feet. | | h m | Feet. | |
| 0 24 | 22 | 22 | 0 26 | 22.73 | 19 | 0 23.5 | 22.87 | 26 | 0 17.5 | 22.8 | 23 | 0 12 | 22.75 | 23 | 0 16 | 22.47 | 19 |
| 0 54.5 | 22.33 | 24 | 0 58 | 22.69 | 25 | 0 53 | 22.8 | 26 | 0 47 | 23.03 | 20 | 0 43.5 | 22.76 | 21 | 0 44 | 22.4 | 20 |
| 1 24 | 22.42 | 20 | 1 27 | 22.75 | 25 | 1 23.5 | 23 | 22 | 1 15 | 22.97 | 23 | 1 12.5 | 22.95 | 21 | 1 14 | 22.64 | 21 |
| 1 55 | 22.79 | 24 | 1 59 | 22.7 | 21 | 1 53.5 | 22.77 | 28 | 1 46 | 22.85 | 23 | 1 42 | 22.89 | 22 | 1 46 | 22.5 | 23 |
| 2 24 | 22.6 | 27 | 2 28.5 | 22.98 | 25 | 2 24 | 22.79 | 23 | 2 16.5 | 23 | 22 | 2 11 | 22.98 | 22 | 2 16 | 22.58 | 20 |
| 2 55 | 22.71 | 22 | 3 0 | 22.72 | 25 | 2 55 | 22.42 | 25 | 2 47 | 22.83 | 20 | 2 42 | 22.52 | 21 | 2 45 | 22.5 | 22 |
| 3 24 | 22.46 | 27 | 3 30 | 22.39 | 24 | 3 25 | 22.5 | 24 | 3 14 | 22.42 | 19 | 3 10.5 | 22.45 | 20 | 3 14 | 22.18 | 22 |
| 3 54 | 22.19 | 27 | 3 59 | 22.23 | 20 | 3 55.5 | 22.4 | 22 | 3 43 | 22.17 | 19 | 3 42 | 22.09 | 26 | 3 45 | 22.1 | 24 |
| 4 26 | 21.92 | 24 | 4 28 | 21.81 | 23 | 4 26 | 21.83 | 24 | 4 13 | 21.83 | 23 | 4 11 | 21.78 | 22 | 4 15 | 22 | 23 |
| 4 55 | 21.7 | 25 | 5 3 | 21.3 | 25 | 4 56 | 21.44 | 20 | 4 43 | 21.5 | 19 | 4 42 | 21.33 | 21 | 4 46 | 21.36 | 26 |
| 5 25 | 21.06 | 26 | 5 31.5 | 21.17 | 22 | 5 24 | 21.17 | 20 | 5 11.5 | 21.02 | 21 | 5 11 | 20.7 | 25 | 5 17 | 21.02 | 24 |
| 5 56 | 21 | 28 | 6 0 | 20.46 | 19 | 5 26 | 20.56 | 24 | 5 44 | 20.28 | 22 | 5 43 | 20.13 | 24 | 5 46 | 20.68 | 23 |
| 6 26 | 20.15 | 23 | 6 29 | 19.6 | 21 | 6 24 | 19.92 | 21 | 6 16 | 19.88 | 25 | 6 10 | 19.98 | 20 | 6 16 | 20.47 | 25 |
| 6 55 | 19.85 | 24 | 7 0.5 | 19.37 | 19 | 6 55 | 19.3 | 23 | 6 47 | 19.08 | 20 | 6 39 | 19.83 | 26 | 6 46 | 20.2 | 27 |
| 7 24 | 19.73 | 24 | 7 29 | 18.5 | 19 | 7 25.5 | 19.3 | 21 | 7 16.5 | 18.92 | 24 | 7 11 | 19.74 | 28 | 7 15 | 19.73 | 22 |
| 7 56 | 19.25 | 25 | 7 58 | 19.1 | 18 | 7 55 | 18.81 | 23 | 7 49 | 18.94 | 23 | 7 41.5 | 19.74 | 14 | 7 46 | 19.86 | 27 |
| 8 26 | 19.85 | 21 | 8 28 | 18.94 | 22 | 8 24.5 | 19.25 | 23 | 8 15.5 | 19.25 | 21 | 8 12.5 | 20.15 | 26 | 8 17 | 20.06 | 23 |
| 8 56 | 19.89 | 23 | 8 58 | 19.3 | 16 | 8 55 | 19.3 | 22 | 8 41.5 | 19.48 | 23 | 8 44 | 20.28 | 27 | 8 46 | 20.56 | 23 |
| 9 27 | 20.23 | 20 | 9 28.5 | 19.85 | 23 | 9 24 | 20.06 | 22 | 9 15 | 20.4 | 26 | 9 12.5 | 20.99 | 21 | 9 16 | 20.7 | 23 |
| 9 54 | 20.37 | 23 | 9 59.5 | 20.2 | 18 | 9 54 | 20.46 | 25 | 9 45 | 21 | 24 | 9 40.5 | 21.19 | 24 | 9 46 | 21.08 | 19 |
| 10 25 | 21.08 | 18 | 10 29 | 20.97 | 21 | 10 24 | 21.04 | 22 | 10 16 | 21.51 | 24 | 10 11 | 21.72 | 25 | 10 15 | 21.28 | 23 |
| 10 53 | 21.44 | 21 | 11 0.5 | 21.45 | 22 | 10 54 | 21.52 | 25 | 10 45 | 22 | 25 | 10 40.5 | 22.28 | 22 | 10 45 | 21.75 | 20 |
| 11 24 | 21.5 | 24 | 11 30 | 21.21 | 18 | 11 25 | 22.12 | 27 | 11 15.5 | 22.69 | 22 | 11 11 | 22.39 | 25 | 11 14 | 22.12 | 19 |
| 11 56 | 22.35 | 21 | 11 59 | 22.22 | 21 | 11 54.5 | 22.34 | 21 | 11 46 | 22.56 | 26 | 11 41 | 22.75 | 11 | 11 45 | 22.14 | 22 |

| July. | | | August. | | | September. | | | October. | | | November. | | | December. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | Height of Tide. | No. of Obs. |
| h m | Feet. | | h m | Feet. | | h m | Feet. | | h m | Feet. | | h m | Feet. | | h m | Feet. | |
| 0 21 | 22.3 | 19 | 0 19 | 22.28 | 22 | 0 9 | 22.67 | 26 | 0 0 | 22.97 | 24 | 23 59 | 22.67 | 22 | 0 13 | 22.75 | 22 |
| 0 49 | 22.43 | 22 | 0 51.5 | 22.5 | 25 | 0 40 | 22.77 | 25 | 0 30 | 23 | 23 | 0 29 | 23.08 | 20 | 0 42 | 22.11 | 19 |
| 1 20 | 22.57 | 23 | 1 19 | 22.73 | 20 | 1 10.5 | 22.92 | 24 | 1 1 | 23.12 | 25 | 0 59 | 23.04 | 22 | 1 9 | 22.26 | 20 |
| 1 50 | 22.72 | 23 | 1 48 | 22.57 | 26 | 1 41 | 22.83 | 24 | 1 31 | 23.11 | 23 | 1 31 | 23 | 21 | 1 40 | 22.46 | 22 |
| 2 20 | 22.69 | 25 | 2 19 | 22.78 | 25 | 2 10 | 23.13 | 20 | 2 2 | 22.97 | 21 | 2 1 | 22.92 | 20 | 2 11 | 22.86 | 23 |
| 2 51 | 22.6 | 25 | 2 48 | 22.75 | 27 | 2 39.5 | 22.91 | 27 | 2 32 | 22.58 | 25 | 2 31 | 22.53 | 22 | 2 41 | 22.46 | 22 |
| 3 21 | 22.5 | 24 | 3 20.5 | 22.59 | 24 | 3 10 | 22.82 | 20 | 3 3 | 22.58 | 20 | 3 1 | 22.8 | 20 | 3 11 | 22.39 | 23 |
| 3 51 | 22.44 | 27 | 3 49.5 | 22.57 | 23 | 3 39 | 22.69 | 22 | 3 31 | 22.62 | 20 | 3 31 | 22.35 | 22 | 3 41 | 22.01 | 24 |
| 4 21 | 22.22 | 23 | 4 19 | 21.6 | 25 | 4 9 | 22.13 | 21 | 4 0 | 21.94 | 21 | 4 3 | 21.9 | 23 | 4 11.5 | 21.95 | 26 |
| 4 50 | 21.75 | 26 | 4 50 | 21.76 | 25 | 4 40 | 21.7 | 25 | 4 29 | 21.61 | 22 | 4 33 | 21.52 | 21 | 4 42 | 21.43 | 25 |
| 5 21 | 21.4 | 28 | 5 20.5 | 21.29 | 22 | 5 12.5 | 21.08 | 21 | 5 1 | 20.84 | 24 | 5 0 | 20.9 | 20 | 5 11.5 | 21.23 | 24 |
| 5 52 | 21 | 25 | 5 49 | 20.99 | 22 | 5 41.5 | 20.7 | 19 | 5 31 | 20.64 | 22 | 5 29 | 20.6 | 23 | 5 41 | 20.78 | 28 |
| 6 21 | 20.67 | 22 | 6 18 | 20.28 | 22 | 6 10.5 | 20.08 | 21 | 6 1 | 19.69 | 20 | 5 59 | 19.9 | 23 | 6 12 | 20.28 | 26 |
| 6 50 | 20.29 | 27 | 6 49 | 19.92 | 24 | 6 41 | 19.62 | 22 | 6 32 | 19.85 | 25 | 6 31 | 19.52 | 26 | 6 42 | 20.04 | 24 |
| 7 21 | 19.97 | 25 | 7 19 | 19.66 | 21 | 7 11.5 | 18.94 | 21 | 7 4 | 19.22 | 23 | 7 1 | 19.17 | 24 | 7 11 | 19.68 | 26 |
| 7 53 | 20.04 | 24 | 7 48 | 19.39 | 22 | 7 42 | 19.23 | 22 | 7 31.5 | 19.04 | 21 | 7 29 | 19.57 | 23 | 7 41 | 19.54 | 23 |
| 8 23 | 20.02 | 21 | 8 19 | 19.31 | 24 | 8 13 | 18.84 | 21 | 7 59.5 | 19.14 | 22 | 8 1 | 19.63 | 26 | 8 10 | 19.74 | 25 |
| 8 51 | 20.23 | 20 | 8 50 | 19.63 | 23 | 8 41 | 19.58 | 22 | 8 29 | 19.82 | 25 | 8 31 | 19.65 | 24 | 8 43 | 20.5 | 26 |
| 9 23 | 20.28 | 23 | 9 20.5 | 19.89 | 22 | 9 11.5 | 19.94 | 23 | 8 59.5 | 20.18 | 27 | 9 0 | 20.65 | 25 | 9 11 | 20.79 | 22 |
| 9 53 | 20.6 | 21 | 9 50 | 20.35 | 21 | 9 42 | 20.42 | 23 | 9 31 | 21.19 | 24 | 9 30 | 20.73 | 25 | 9 41 | 21.3 | 23 |
| 10 23 | 20.9 | 21 | 10 18 | 20.73 | 20 | 10 12 | 21 | 23 | 10 0 | 21.38 | 26 | 10 0 | 21.17 | 23 | 10 12 | 21.28 | 23 |
| 10 53 | 21.28 | 20 | 10 48 | 21.14 | 22 | 10 41.5 | 21.64 | 23 | 10 30 | 22.08 | 26 | 10 31 | 21.83 | 25 | 10 41.5 | 21.62 | 20 |
| 11 23 | 21.67 | 23 | 11 18 | 21.73 | 23 | 11 9 | 22.08 | 21 | 11 1 | 22.19 | 25 | 11 1 | 22.47 | 22 | 11 10 | 21.77 | 21 |
| 11 53 | 21.84 | 21 | 11 48.5 | 22.03 | 23 | 11 40 | 22.31 | 17 | 11 31 | 22.48 | 25 | 11 30 | 22.58 | 21 | 11 39 | 22.06 | 20 |

## Table V. (Interpolated from Table IV.)
## Showing the Height of High Water at the London Docks.

| | Jan. | Feb. | March. | April. | May. | June. | July. | Aug. | Sept. | Oct. | Nov. | Dec. | Mean. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | Height of Tide. | B. |
| h m | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. |
| 0 0 | 22.30 | 22.23 | 22.44 | 22.67 | 22.75 | 22.30 | 21.95 | 22.13 | 22.56 | 22.97 | 22.68 | 22.49 | 22.46 |
| 0 30 | 22.07 | 22.73 | 22.85 | 22.90 | 22.75 | 22.43 | 22.34 | 22.35 | 22.74 | 23.00 | 23.08 | 22.37 | 22.63 |
| 1 0 | 22.36 | 22.70 | 22.85 | 23.00 | 22.87 | 22.53 | 22.48 | 22.56 | 22.87 | 23.12 | 23.04 | 22.21 | 22.72 |
| 1 30 | 22.47 | 22.75 | 22.95 | 22.91 | 22.91 | 22.57 | 22.62 | 22.67 | 22.86 | 23.11 | 23.00 | 22.39 | 22.76 |
| 2 0 | 22.76 | 22.71 | 22.78 | 22.92 | 22.95 | 22.54 | 22.71 | 22.65 | 23.03 | 22.98 | 22.92 | 22.72 | 22.80 |
| 2 30 | 22.62 | 22.97 | 22.72 | 22.93 | 22.70 | 22.54 | 22.66 | 22.77 | 22.96 | 22.60 | 22.54 | 22.61 | 22.72 |
| 3 0 | 22.67 | 22.72 | 22.43 | 22.63 | 22.47 | 22.34 | 22.57 | 22.69 | 22.85 | 22.58 | 22.79 | 22.35 | 22.59 |
| 3 30 | 22.41 | 22.39 | 22.48 | 22.28 | 22.23 | 22.14 | 22.48 | 22.49 | 22.73 | 22.62 | 22.36 | 22.11 | 22.39 |
| 4 0 | 22.14 | 22.22 | 22.29 | 21.98 | 21.89 | 22.05 | 22.37 | 22.08 | 22.30 | 21.94 | 21.94 | 21.97 | 22.10 |
| 4 30 | 21.89 | 21.79 | 21.78 | 21.64 | 21.49 | 21.69 | 22.07 | 21.66 | 21.84 | 21.61 | 21.56 | 21.64 | 21.72 |
| 5 0 | 21.59 | 21.34 | 21.40 | 21.22 | 20.94 | 21.21 | 21.63 | 21.60 | 21.32 | 20.86 | 20.90 | 21.31 | 21.28 |
| 5 30 | 21.05 | 21.18 | 21.11 | 20.59 | 20.37 | 20.86 | 21.28 | 21.19 | 20.85 | 20.65 | 20.58 | 20.95 | 20.88 |
| 6 0 | 20.89 | 20.46 | 20.46 | 20.08 | 20.03 | 20.58 | 20.91 | 20.72 | 20.30 | 19.72 | 19.89 | 20.47 | 20.37 |
| 6 30 | 20.11 | 19.59 | 19.80 | 19.52 | 19.87 | 20.34 | 20.55 | 20.14 | 19.79 | 19.84 | 19.53 | 20.14 | 19.93 |
| 7 0 | 19.83 | 19.37 | 19.30 | 19.01 | 19.79 | 19.98 | 20.19 | 19.82 | 19.20 | 19.30 | 19.18 | 19.82 | 19.56 |
| 7 30 | 19.64 | 18.68 | 19.23 | 18.93 | 19.74 | 19.79 | 19.99 | 19.56 | 19.11 | 19.05 | 19.57 | 19.59 | 19.40 |
| 8 0 | 19.33 | 19.09 | 18.88 | 19.07 | 19.98 | 19.95 | 20.03 | 19.36 | 19.00 | 19.15 | 19.63 | 19.67 | 19.43 |
| 8 30 | 19.86 | 18.96 | 19.26 | 19.38 | 20.22 | 20.28 | 20.07 | 19.42 | 19.29 | 19.83 | 19.65 | 20.20 | 19.70 |
| 9 0 | 19.93 | 19.32 | 19.43 | 19.98 | 20.68 | 20.62 | 20.24 | 19.72 | 19.80 | 20.20 | 20.65 | 20.68 | 20.10 |
| 9 30 | 20.24 | 19.86 | 20.14 | 20.70 | 21.12 | 20.88 | 20.35 | 20.04 | 20.23 | 21.16 | 20.73 | 21.11 | 20.54 |
| 10 0 | 20.51 | 20.20 | 20.58 | 21.25 | 21.53 | 21.18 | 20.67 | 20.48 | 20.81 | 21.38 | 21.17 | 21.29 | 20.92 |
| 10 30 | 21.14 | 20.98 | 21.14 | 21.75 | 22.08 | 21.51 | 20.99 | 20.89 | 21.39 | 22.08 | 21.81 | 21.49 | 21.43 |
| 11 0 | 21.45 | 21.45 | 21.64 | 22.34 | 22.35 | 21.94 | 21.37 | 21.37 | 21.94 | 22.19 | 22.45 | 21.72 | 21.85 |
| 11 30 | 21.66 | 21.21 | 22.16 | 22.63 | 22.63 | 22.11 | 21.71 | 21.85 | 22.24 | 22.47 | 22.58 | 21.97 | 22.10 |
| | 21.29 | 21.12 | 21.25 | 21.34 | 21.51 | 21.43 | 21.43 | 21.26 | 21.33 | 21.43 | 21.42 | 21.39 | 21.34 |

## Table VI.

Showing the Time and the Height of High Water at the London Docks, corresponding to the mean time of the Moon's Transit for every minute of Horizontal Parallax, from 5414 observations made between the 1st of January 1808 and 31st of December 1826.

| Hor. Par. 54′. | | | | Hor. Par. 55′. | | | | Hor. Par. 56′. | | | | Hor. Par. 57′. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. |
| h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | |
| 0 34 | 2 34.9 | 22 1 | 95 | 0 31.5 | 2 29.8 | 22 3.4 | 51 | 0 34 | 2 34.9 | 22 4.3 | 38 | 0 33.8 | 2 25.1 | 22 7.7 | 43 |
| 1 19.7 | 3 6.4 | 22 2.6 | 89 | 1 21.2 | 3 3.8 | 22 3.6 | 54 | 1 22.4 | 3 1.8 | 22 9.8 | 50 | 1 26 | 3 7 | 22 8.3 | 38 |
| 2 35.8 | 3 59.2 | 22 1.4 | 80 | 2 37.4 | 3 58.1 | 22 6.5 | 71 | 2 31.8 | 3 51 | 22 5.2 | 38 | 2 33.3 | 3 51.2 | 22 10 | 45 |
| 3 22.3 | 4 33.2 | 21 8.5 | 73 | 3 23.1 | 4 31.2 | 22 1.3 | 65 | 3 23 | 4 30 | 22 2.3 | 56 | 3 20.7 | 4 27 | 22 5.1 | 46 |
| 4 29.3 | 5 20.2 | 21 1 | 72 | 4 31.5 | 5 23.8 | 21 3.5 | 74 | 4 32.2 | 5 26.2 | 21 4.8 | 50 | 4 33 | 5 22.1 | 21 9.3 | 59 |
| 5 34.3 | 6 18 | 20 2.2 | 55 | 5 35.4 | 6 18 | 20 3 | 90 | 5 31.2 | 6 13.2 | 20 4 | 58 | 5 33.6 | 6 18 | 21 0.7 | 57 |
| 6 34.8 | 7 22 | 19 3.1 | 68 | 6 38 | 7 27.3 | 19 6.8 | 71 | 6 36.5 | 7 18.6 | 19 8.3 | 66 | 6 33.7 | 7 17.7 | 19 10.2 | 57 |
| 7 25.8 | 8 35 | 18 8.5 | 72 | 7 31.1 | 8 34.5 | 19 0.1 | 73 | 7 26.3 | 8 27.7 | 19 2.2 | 58 | 7 24.8 | 8 23.5 | 19 6 | 56 |
| 8 29.7 | 10 30 | 19 3.2 | 86 | 8 37.6 | 10 29 | 19 8 | 54 | 8 35 | 10 20 | 19 7.2 | 65 | 8 35.8 | 10 19.2 | 19 10.2 | 44 |
| 9 38.5 | 12 2.4 | 20 2.2 | 82 | 9 38.5 | 11 56.3 | 20 5.5 | 66 | 9 35.5 | 11 49.2 | 20 5.7 | 42 | 9 35.5 | 11 43 | 20 4.3 | 44 |
| 10 54.7 | 13 5 | 21 3 | 84 | 10 39.2 | 13 1.9 | 21 0.7 | 56 | 10 35.7 | 12 53.6 | 21 4 | 50 | 10 33.6 | 12 43 | 21 5 | 39 |
| 11 36.5 | 13 51.6 | 21 8 | 83 | 11 35.3 | 13 46.8 | 21 10.5 | 60 | 11 33 | 13 41.7 | 21 9.7 | 35 | 11 38 | 13 18.6 | 22 4.5 | 43 |

| Hor. Par. 58′. | | | | Hor. Par. 59′. | | | | Hor. Par. 60′. | | | | Hor. Par. 61′. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. P.M. | High Water. | Height of Tide. | No. of Obs. |
| h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | |
| 0 36.2 | 2 21.8 | 22 9.3 | 42 | 0 37.6 | 2 20.9 | 23 0 | 33 | 0 31.8 | 2 12.5 | 22 11.1 | 52 | 0 32.2 | 2 10.4 | 23 1.5 | 92 |
| 1 27.3 | 3 2 | 23 0 | 37 | 1 28 | 3 1 | 23 0 | 42 | 1 30.8 | 2 56 | 23 2.1 | 66 | 1 28.2 | 2 53.2 | 23 6.5 | 78 |
| 2 33.6 | 3 51.8 | 22 9.7 | 50 | 2 30 | 3 46 | 23 0.9 | 48 | 2 34 | 3 45.4 | 23 5.2 | 72 | 2 28.3 | 3 39.3 | 23 6.3 | 43 |
| 3 24 | 4 30.8 | 22 5.2 | 46 | 3 24.6 | 4 26.3 | 22 10.4 | 73 | 3 28.1 | 4 26.4 | 23 3.4 | 90 | 3 29 | 4 32.5 | 21 2.5 | 2 |
| 4 33.3 | 5 22.3 | 22 2.6 | 55 | 4 32.3 | 5 20.6 | 22 3.7 | 114 | 4 33.7 | 5 8 | 22 5.4 | 27 | | | | |
| 5 31.7 | 6 13 | 21 2.4 | 73 | 5 31.3 | 6 12.8 | 21 6.8 | 119 | | | | | | | | |
| 6 30.4 | 7 11 | 20 3 | 57 | 6 34 | 7 15.6 | 20 5.2 | 121 | 6 33.2 | 7 22 | 20 3.2 | 8 | | | | |
| 7 28.2 | 8 25.4 | 19 7.8 | 64 | 7 26.6 | 8 24 | 19 10.1 | 88 | 7 36 | 8 36.4 | 20 0.6 | 40 | | | | |
| 8 31.2 | 10 4 | 19 7.2 | 49 | 8 32 | 10 6.7 | 20 2.8 | 56 | 8 34.8 | 10 4.2 | 20 3.5 | 99 | | | | |
| 9 40 | 11 40 | 20 7.5 | 46 | 9 35.5 | 11 30 | 20 11 | 51 | 9 34.6 | 11 23.2 | 21 0.7 | 98 | 9 26.3 | 11 27 | 21 5 | 22 |
| 10 33.2 | 12 41 | 21 8.2 | 43 | 10 37.2 | 12 37 | 21 9 | 42 | 10 33.8 | 12 29.8 | 21 10.3 | 63 | 10 32.3 | 12 25.2 | 22 1.8 | 73 |
| 11 38.8 | 13 38 | 22 2.8 | 39 | 11 31.6 | 13 30 | 22 2.2 | 43 | 11 35.9 | 13 22 | 22 6.5 | 50 | 11 34.7 | 13 20.7 | 22 9 | 97 |

## Table VII. (Interpolated from Table VI.)

Showing the Interval between the Moon's Transit and the Time of High Water at the London Docks, for every minute of her Horizontal Parallax.

| Moon's Transit. | H. P. 54′. | H. P. 55′. | H. P. 56′. | H. P. 57′. | H. P. 58′. | H. P. 59′. | H. P. 60′. | H. P. 61′. |
|---|---|---|---|---|---|---|---|---|
| h m | h m | h m | h m | h m | h m | h m | h m | h m |
| 0 0 | 2 9 | 2 5.8 | 2 1 | 1 56.6 | 1 53.9 | 1 53 | 1 43.5 | 1 42.6 |
| 0 30 | 2 1.9 | 1 58.8 | 1 51.6 | 1 52 | 1 47 | 1 44.9 | 1 41.1 | 1 38.6 |
| 1 0 | 1 52.6 | 1 49.7 | 1 45 | 1 46.8 | 1 40.8 | 1 38.6 | 1 33.4 | 1 31.5 |
| 1 30 | 1 43.4 | 1 39.7 | 1 37.6 | 1 40 | 1 34.1 | 1 32.5 | 1 25.4 | 1 24.4 |
| 2 0 | 1 34.2 | 1 31.8 | 1 28.6 | 1 29.6 | 1 27 | 1 25 | 1 18.6 | 1 17 |
| 2 30 | 1 25 | 1 23 | 1 19.7 | 1 19 | 1 19.8 | 1 16 | 1 12.4 | 1 10.7 |
| 3 0 | 1 16.9 | 1 14.8 | 1 12.3 | 1 10.5 | 1 12.6 | 1 8.3 | 1 5.2 | |
| 3 30 | 1 8.8 | 1 6.8 | 1 6 | 1 4 | 1 6.3 | 1 0.4 | 0 57.4 | |
| 4 0 | 0 59.8 | 0 59.2 | 0 58.6 | 0 55.6 | 0 58 | 0 54 | 0 50 | |
| 4 30 | 0 50.8 | 0 52.4 | 0 54.2 | 0 49.6 | 0 50.1 | 0 48.6 | 0 42.7 | |
| 5 0 | 0 46.3 | 0 46.6 | 0 47.6 | 0 46 | 0 45 | 0 44 | | |
| 5 30 | 0 44 | 0 42.7 | 0 42 | 0 44.5 | 0 41.4 | 0 41.8 | | |
| 6 0 | 0 43 | 0 43.4 | 0 39.5 | 0 43.4 | 0 39.8 | 0 40.4 | | |
| 6 30 | 0 46.2 | 0 47.9 | 0 41 | 0 43.6 | 0 41 | 0 41.2 | | |
| 7 0 | 0 56.2 | 0 53.7 | 0 49 | 0 49.6 | 0 47 | 0 47 | | |
| 7 30 | 1 14.6 | 1 3.1 | 1 3.3 | 1 1.5 | 0 58 | 0 58.8 | 0 58.4 | |
| 8 0 | 1 38.2 | 1 22 | 1 22 | 1 18.8 | 1 14.8 | 1 16.6 | 1 11.6 | |
| 8 30 | 2 0.5 | 1 47 | 1 42.6 | 1 40.2 | 1 39 | 1 33.8 | 1 27.6 | |
| 9 0 | 2 14.6 | 2 4.8 | 2 0 | 1 56.6 | 1 46.4 | 1 45.2 | 1 39.2 | |
| 9 30 | 2 22.3 | 2 15.7 | 2 12.5 | 2 6.3 | 1 57.2 | 1 53.5 | 1 47.5 | |
| 10 0 | 2 26 | 2 21.5 | 2 17.6 | 2 10 | 2 4.8 | 1 58 | 1 53.2 | |
| 10 30 | 2 25.4 | 2 23.2 | 2 18.2 | 2 9.6 | 2 7.8 | 2 0 | 1 56 | 1 53.2 |
| 11 0 | 2 21.8 | 2 20 | 2 15.5 | 2 6.2 | 2 6 | 2 0.5 | 1 54.3 | 1 50 |
| 11 30 | 2 16.2 | 2 12.7 | 2 9.4 | 2 1.8 | 2 0.8 | 1 58.8 | 1 47 | 1 [illegible] |

## Table VIII. (Interpolated from Table VI.)

Showing the Height of High Water at the London Docks for every minute of the Moon's Horizontal Parallax.

| Moon's Transit. | H. P. 54′. | H. P. 55′. | H. P. 56′. | H. P. 57′. | H. P. 58′. | H. P. 59′. | H. P. 60′. | H. P. 61′. |
|---|---|---|---|---|---|---|---|---|
| h m | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. |
| 0 0 | 21.85 | 22.04 | 22.07 | 22.47 | 22.43 | 22.60 | 22.70 | 22.90 |
| 0 30 | 22.06 | 22.28 | 22.36 | 22.64 | 22.73 | 22.97 | 22.92 | 23.09 |
| 1 0 | 22.16 | 22.30 | 22.65 | 22.67 | 22.90 | 22.98 | 23.05 | 23.35 |
| 1 30 | 22.22 | 22.31 | 22.80 | 22.70 | 23.00 | 23.00 | 23.18 | 23.54 |
| 2 0 | 22.24 | 22.29 | 22.66 | 22.76 | 22.93 | 23.05 | 23.31 | 23.56 |
| 2 30 | 22.19 | 22.28 | 22.45 | 22.83 | 22.81 | 23.07 | 23.42 | 23.52 |
| 3 0 | 21.87 | 22.18 | 22.30 | 22.61 | 22.60 | 22.97 | 23.38 | |
| 3 30 | 21.64 | 22.04 | 22.14 | 22.35 | 22.40 | 22.83 | 23.27 | |
| 4 0 | 21.37 | 21.66 | 21.82 | 22.08 | 22.14 | 22.60 | 22.92 | |
| 4 30 | 21.08 | 21.28 | 21.44 | 21.80 | 22.24 | 22.53 | 22.40 | |
| 5 0 | 20.70 | 20.80 | 20.90 | 21.45 | 21.73 | 22.00 | | |
| 5 30 | 20.25 | 20.32 | 20.35 | 21.09 | 21.22 | 21.58 | | |
| 6 0 | 19.78 | 19.95 | 20.04 | 20.50 | 20.73 | 21.00 | | |
| 6 30 | 19.31 | 19.64 | 19.76 | 19.90 | 20.25 | 20.47 | 20.28 | |
| 7 0 | 18.95 | 19.30 | 19.48 | 19.64 | 19.90 | 20.13 | 20.12 | |
| 7 30 | 18.75 | 19.02 | 19.22 | 19.51 | 19.65 | 19.83 | 20.08 | |
| 8 0 | 18.90 | 19.28 | 19.33 | 19.63 | 19.60 | 19.50 | 20.12 | |
| 8 30 | 19.27 | 19.54 | 19.55 | 19.82 | 19.61 | 20.23 | 20.25 | |
| 9 0 | 19.90 | 19.93 | 19.95 | 20.05 | 19.93 | 19.40 | 20.55 | |
| 9 30 | 20.58 | 20.31 | 20.40 | 20.29 | 20.46 | 20.85 | 21.00 | 21.42 |
| 10 0 | 20.85 | 20.64 | 20.83 | 20.80 | 21.08 | 21.30 | 21.43 | 21.63 |
| 10 30 | 21.13 | 20.96 | 21.26 | 21.36 | 21.65 | 21.69 | 21.81 | 21.88 |
| 11 0 | 21.38 | 21.38 | 21.55 | 21.90 | 21.98 | 21.95 | 22.20 | 22.30 |
| 11 30 | 21.60 | 21.81 | 21.78 | 22.30 | 22.16 | 22.17 | 22.49 | 22.71 |

## Table IX.

Showing the Time and Height of High Water at the London Docks corresponding to the mean time of the Moon's Transit for every three degrees of her declination north and south; from 5372 observations made between the 1st of January 1808 and the 31st of December 1826.

| 1° 30′ S. to 1° 30′ N. Decl. | | | |
|---|---|---|---|
| Moon's Transit. | High Water. | Height of Tide. | No. of Obs. |
| h m | h m | ft. in. | |
| 0 35 | 2 32.5 | 22 9.5 | 22 |
| 1 26 | 3 7 | 23 1 | 21 |
| 2 42 | 4 4 | 23 1 | 17 |
| 3 26 | 4 40.5 | 22 7.8 | 22 |
| 4 35 | 5 33 | 21 10 | 18 |
| 5 36 | 6 36.7 | 21 0 | 21 |
| 6 36.5 | 7 43.5 | 19 9 | 17 |
| 7 22 | 8 40.5 | 19 11 | 17 |
| 8 32 | 10 27 | 19 11 | 23 |
| 9 39 | 11 59 | 21 4 | 13 |
| 10 31 | 12 52 | 21 9 | 21 |
| 11 36 | 13 46 | 22 3 | 21 |

| 1° 30′ to 4° 30′ North Dec. | | | | 4° 30′ to 7° 30′ North Dec. | | | | 7° 30′ to 10° 30′ North Dec. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. | High Water. | Height of Tide. | No. of Obs. |
| h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | |
| 0 38 | 2 35 | 22 8.5 | 19 | 0 36 | 2 37.5 | 22 8 | 20 | 0 41 | 2 36 | 22 6 | 21 |
| 1 27 | 3 12 | 22 9 | 18 | 1 30 | 3 11 | 22 11 | 23 | 1 27 | 3 6 | 22 11 | 19 |
| 2 41 | 4 2 | 22 8 | 19 | 2 42 | 4 3 | 22 8 | 20 | 2 41 | 4 3 | 23 2 | 26 |
| 3 28 | 4 41 | 22 6 | 22 | 3 26 | 4 42 | 22 2 | 15 | 3 22 | 4 41 | 22 4.7 | 16 |
| 4 37 | 5 37 | 21 8.5 | 19 | 4 42 | 5 38 | 22 | 19 | 4 37 | 5 37 | 21 9.3 | 20 |
| 5 32 | 6 31 | 20 5 | 10 | 5 34 | 6 28 | 21 4.5 | 15 | 5 41 | 6 40 | 20 7.5 | 12 |
| 6 36.5 | 7 46 | 20 3.5 | 20 | 6 31 | 7 27.5 | 20 2 | 14 | 6 34.5 | 7 34.5 | 20 | 19 |
| 7 19 | 8 32.5 | 20 1 | 18 | 7 20.2 | 8 34 | 19 9 | 19 | 7 22.5 | 8 43 | 19 11.5 | 18 |
| 8 28 | 10 17 | 20 4 | 17 | 8 31 | 10 21 | 19 11 | 21 | 8 26 | 10 15 | 19 9 | 20 |
| 9 29 | 11 37 | 21 1 | 18 | 9 35 | 11 52 | 20 8 | 22 | 9 32 | 11 46 | 20 8 | 25 |
| 10 35 | 12 56 | 21 1 | 15 | 10 38 | 12 53 | 22 | 23 | 10 35 | 12 54 | 22 | 31 |
| 11 28 | 13 32 | 22 4 | 14 | 11 38 | 13 44 | 22 4 | 18 | 11 33 | 13 41 | 22 5 | 15 |

| 1° 30′ to 4° 30′ South Dec. | | | | 4° 30′ to 7° 30′ South Dec. | | | | 7° 30′ to 10° 30′ South Dec. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 36 | 2 29 | 22 7 | 18 | 0 34 | 2 28 | 22 9 | 17 | 0 33 | 2 28 | 22 8.5 | 21 |
| 1 28 | 3 10 | 22 6 | 18 | 1 30 | 3 13 | 22 11 | 20 | 1 30 | 3 6 | 22 9 | 23 |
| 2 39 | 4 4.7 | 22 3 | 25 | 2 39 | 4 0 | 22 11 | 16 | 2 36 | 4 0 | 22 10 | 17 |
| 3 30 | 4 46 | 22 6.5 | 16 | 3 21 | 4 32 | 22 8 | 15 | 3 20 | 4 33 | 22 4 | 23 |
| 4 35 | 5 33 | 21 5.5 | 18 | 4 39 | 5 39 | 21 8 | 15 | 4 34 | 5 31 | 21 9 | 14 |
| 5 29 | 6 22.7 | 21 1.7 | 15 | 5 40 | 6 36 | 21 4 | 15 | 5 32 | 6 24 | 21 0.3 | 20 |
| 6 37 | 7 35 | 20 2.5 | 18 | 6 40.5 | 7 40 | 20 2 | 17 | 6 36 | 7 34 | 20 4 | 19 |
| 7 18.5 | 8 43.5 | 19 6.5 | 17 | 7 19 | 8 42 | 19 7 | 17 | 7 20 | 8 38 | 20 2 | 20 |
| 8 24 | 10 21 | 20 5 | 13 | 8 37 | 10 39 | 20 5 | 24 | 8 36 | 10 31 | 20 1 | 19 |
| 9 37 | 11 53 | 21 2 | 18 | 9 30 | 11 45 | 20 10 | 14 | 9 36 | 11 47 | 21 1 | 24 |
| 10 45 | 12 59 | 22 1 | 17 | 10 31 | 12 42 | 21 10 | 20 | 10 35 | 12 49 | 21 10 | 18 |
| 11 39 | 13 46 | 22 7 | 20 | 11 31 | 13 39 | 22 5 | 26 | 11 42 | 13 45 | 22 5 | 19 |

## Table IX. (Continued).

| 10° 30′ to 13° 30′ North Dec. | | | | 13° 30′ to 16° 30′ North Dec. | | | | 16° 30′ to 19° 30′ North Dec. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Moon's Transit. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. | High Water. | Height of Tide. | No. of Obs. | Moon's Transit. | High Water. | Height of Tide. | No. of Obs. |
| h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | |
| 0 38 | 2 25 | 22 10 | 17 | 0 37 | 2 27 | 22 6.7 | 28 | 0 32 | 2 21 | 22 7.2 | 37 |
| 1 30 | 3 8 | 22 8 | 27 | 1 28 | 3 8 | 22 9 | 23 | 1 23 | 2 58 | 22 11 | 33 |
| 2 39 | 3 56 | 22 11 | 25 | 2 37 | 3 57 | 22 7 | 21 | 2 31 | 3 52 | 23 0 | 39 |
| 3 28 | 4 37 | 22 8 | 24 | 3 28 | 4 35 | 22 5.5 | 24 | 3 27 | 4 31 | 22 5.5 | 45 |
| 4 37 | 5 34 | 21 5.5 | 21 | 4 34 | 5 23 | 21 7.5 | 30 | 4 36 | 5 28 | 21 9.5 | 36 |
| 5 35.5 | 6 25 | 20 9 | 26 | 5 38 | 6 18 | 21 2 | 24 | 5 37.5 | 6 13.2 | 20 8.5 | 31 |
| 6 34.5 | 7 24 | 20 0.5 | 26 | 6 36.5 | 7 20.5 | 20 1 | 20 | 6 42 | 7 22.5 | 19 7.7 | 34 |
| 7 30 | 8 41 | 19 6.5 | 21 | 7 28 | 8 37 | 19 3.3 | 27 | 7 37 | 8 38 | 19 1.3 | 40 |
| 8 32 | 10 19 | 20 7 | 26 | 8 35 | 10 20 | 20 1 | 22 | 8 41 | 10 22 | 19 7 | 38 |
| 9 37 | 11 54 | 21 1 | 19 | 9 37 | 11 42 | 20 7 | 31 | 9 42 | 11 45 | 20 5 | 40 |
| 10 34 | 12 46 | 21 9 | 19 | 10 40 | 12 53.5 | 21 8 | 29 | 10 38 | 12 44 | 21 4 | 44 |
| 11 37 | 13 43 | 22 8 | 29 | 11 35 | 13 43 | 21 10 | 28 | 11 37 | 13 39 | 22 2 | 36 |
| 10° 30′ to 13° 30′ South Dec. | | | | 13° 30′ to 16° 30′ South Dec. | | | | 16° 30′ to 19° 30′ South Dec. | | | |
| 0 40.6 | 2 26.4 | 22 8.7 | 25 | 0 32 | 2 20 | 23 0 | 25 | 0 30 | 2 17 | 22 7.5 | 39 |
| 1 21 | 3 2 | 22 9 | 20 | 1 23 | 3 2 | 22 11 | 26 | 1 20 | 2 52 | 22 10 | 38 |
| 2 32 | 3 53 | 23 0 | 24 | 2 26 | 3 45 | 22 10.5 | 28 | 2 29 | 3 42 | 22 11 | 34 |
| 3 20 | 4 28 | 22 6.8 | 20 | 3 25.5 | 4 27 | 22 7 | 29 | 3 21 | 4 22 | 22 9.3 | 35 |
| 4 31 | 5 31 | 21 8.2 | 22 | 4 27 | 5 17 | 21 11.2 | 28 | 4 25 | 5 15 | 21 11 | 39 |
| 5 39 | 6 36.5 | 20 11 | 21 | 5 24 | 6 9 | 20 9 | 28 | 5 28 | 6 10.5 | 21 0.5 | 39 |
| 6 38 | 7 35 | 20 2.3 | 23 | 6 32 | 7 19 | 19 10.3 | 26 | 6 28 | 7 5 | 20 0.2 | 38 |
| 7 20 | 8 35 | 19 7 | 22 | 7 23 | 8 30 | 19 3 | 26 | 7 24 | 8 19 | 19 6 | 42 |
| 8 32 | 10 22 | 20 4 | 24 | 8 36 | 10 18 | 20 1 | 25 | 8 32 | 10 12 | 19 9 | 36 |
| 9 33 | 11 43 | 20 10 | 29 | 9 35 | 11 38 | 20 9 | 22 | 9 36 | 11 35 | 20 8 | 45 |
| 10 35 | 12 50 | 21 8 | 18 | 10 34 | 12 41 | 21 4.5 | 25 | 10 36 | 12 38 | 21 5 | 39 |
| 11 34 | 13 40 | 22 10 | 14 | 11 38 | 13 38 | 22 4 | 25 | 11 32 | 13 30 | 22 2 | 36 |

| 19° 30′ to 22° 30′ North Dec. | | | | 22° 30′ to 25° 30′ North Dec. | | | | Above 25° 30′ North Dec. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| h m | h m | ft. in. | | h m | h m | ft. in. | | h m | h m | ft. in. | |
| 0 38.5 | 2 19 | 22 7 | 26 | 0 31 | 2 16 | 22 6.5 | 25 | 0 31 | 2 13 | 22 5.5 | 17 |
| 1 28 | 2 56 | 22 9 | 30 | 1 24 | 2 57 | 22 9 | 26 | 1 25 | 2 51 | 22 8 | 18 |
| 2 33 | 3 49 | 22 6 | 23 | 2 33 | 3 49 | 22 7 | 24 | 2 29 | 3 40 | 23 0 | 22 |
| 3 25 | 4 25 | 22 5.2 | 24 | 3 23 | 4 19 | 22 3 | 22 | 3 20 | 4 15 | 22 4.5 | 23 |
| 4 32 | 5 14 | 21 9.5 | 28 | 4 35 | 5 14 | 21 4.7 | 26 | 4 27 | 5 5 | 21 8.3 | 23 |
| 5 35 | 6 15 | 20 5 | 30 | 5 35 | 6 4.5 | 20 8.8 | 25 | 5 40.7 | 6 10.2 | 20 8 | 23 |
| 6 37 | 7 10 | 19 9 | 32 | 6 43.5 | 7 9.7 | 19 3 | 22 | 6 35 | 7 1 | 19 1 | 23 |
| 7 34.5 | 8 33.3 | 18 5 | 27 | 7 38 | 8 23 | 19 0.5 | 23 | 7 41 | 8 15 | 18 10 | 22 |
| 8 36 | 10 10 | 19 4 | 19 | 8 41 | 10 3 | 19 5 | 29 | 8 40 | 10 0 | 18 10 | 20 |
| 9 41 | 11 34 | 20 3 | 26 | 9 40 | 11 28.7 | 19 10 | 21 | 9 47 | 11 42 | 20 4 | 19 |
| 10 38 | 12 43 | 21 2 | 20 | 10 39 | 12 42 | 21 4 | 24 | 10 37 | 12 38 | 20 11 | 18 |
| 11 33 | 13 32 | 22 3 | 30 | 11 37 | 13 26 | 21 8 | 17 | 11 33 | 13 25 | 21 9 | 21 |
| 19° 30′ to 22° 30′ South Dec. | | | | 22° 30′ to 25° 30′ South Dec. | | | | Above 25° 30′ South Dec. | | | |
| 0 31 | 2 13 | 22 4.7 | 27 | 0 34 | 2 15 | 22 6.8 | 16 | 0 26 | 2 4 | 22 5 | 22 |
| 1 24 | 2 54 | 23 1 | 23 | 1 17 | 2 47 | 22 8 | 30 | 1 26 | 2 48 | 21 11 | 16 |
| 2 25 | 3 37 | 22 7 | 31 | 2 29 | 3 38 | 22 6 | 21 | 2 21 | 3 32 | 22 6 | 21 |
| 3 22 | 4 20.5 | 22 7.8 | 28 | 3 19 | 4 13.5 | 21 11.3 | 24 | 3 20 | 4 11 | 22 2 | 21 |
| 4 26 | 5 14 | 21 8.7 | 31 | 4 19 | 5 0 | 21 9.5 | 22 | 4 15 | 4 47 | 21 9.5 | 24 |
| 5 23.5 | 6 0.5 | 20 9.5 | 28 | 5 24.4 | 6 0.4 | 20 11.2 | 28 | 5 23.8 | 5 50.3 | 20 6.7 | 26 |
| 6 30.5 | 7 8 | 20 3 | 28 | 6 30 | 6 59 | 19 8 | 22 | 6 23 | 6 43 | 19 9 | 26 |
| 7 20 | 8 15 | 19 4.5 | 23 | 7 24 | 8 11 | 19 4.5 | 24 | 7 25 | 7 58 | 18 5.3 | 23 |
| 8 31 | 10 8 | 19 7 | 28 | 8 39 | 10 2 | 19 4 | 27 | 8 33 | 9 55 | 19 4 | 18 |
| 9 37 | 11 44 | 20 5 | 26 | 9 34 | 11 33 | 20 3 | 20 | 9 33 | 11 29 | 20 1 | 21 |
| 10 39 | 12 43 | 21 3.5 | 28 | 10 34 | 12 40 | 21 3.5 | 27 | 10 39 | 12 32 | 20 10.5 | 19 |
| 11 34.5 | 13 25 | 22 4 | 27 | 11 35 | 13 22 | 21 11 | 25 | 11 35 | 13 25 | 21 8 | 21 |

## Table X. (Interpolated from Table IX.)

Showing the Interval between the Moon's Transit and the Time of High Water, and the Height of High Water at the London Docks for every three degrees of her Declination, North and South.

| Time of Moon's Transit. | 0° Dec. | | 3° Dec. N. | | 6° Dec. N. | | 9° Dec. N. | | 12° Dec. N. | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Interval. | Height. | Interval. | Height. | Interval. | Height. | Interval. | Height. | Interval. | Height. |
| h m | h m | ft. in. | h m | ft. in. | h m | ft. in. | h m | ft. in. | h m | ft. in. |
| 0 30 | 1 58.9 | 22 8.6 | 1 58.4 | 22 7.8 | 2 2.8 | 22 7.7 | 1 58 | 22 5.8 | 1 49 | 22 9.7 |
| 1 30 | 1 40 | 23 1.2 | 1 44 | 22 9 | 1 41 | 22 11 | 1 38.5 | 22 11.2 | 1 38 | 22 8 |
| 2 30 | 1 24.5 | 23 1.3 | 1 24 | 22 8 | 1 23.7 | 22 8.3 | 1 24.7 | 23 1.5 | 1 19.3 | 22 10.5 |
| 3 30 | 1 10.7 | 22 7.3 | 1 13 | 22 6 | 1 15.2 | 22 2.4 | 1 9.4 | 22 5 | 1 8.7 | 22 7.8 |
| 4 30 | 0 59 | 21 10.7 | 1 0 | 21 9.8 | 0 58 | 22 0.2 | 1 0.5 | 21 10.4 | 0 58 | 21 6.8 |
| 5 30 | 1 0.2 | 21 1.2 | 0 59 | 20 5.3 | 0 54 | 21 4.8 | 1 0 | 20 9.7 | 0 50 | 20 9.8 |
| 6 30 | 1 5.1 | 19 10.1 | 1 10.4 | 20 3.7 | 0 56.1 | 20 2 | 0 59.1 | 20 0.3 | 0 49 | 20 0.1 |
| 7 30 | 1 10.5 | 19 11 | 1 11.3 | 20 0.1 | 1 9.3 | 19 9 | 1 24 | 19 11.3 | 1 11 | 19 6.5 |
| 8 30 | 1 55 | 19 11 | 1 49 | 20 4 | 2 0.3 | 19 11 | 1 51 | 19 9.5 | 1 46 | 20 6.7 |
| 9 30 | 2 18.2 | 21 2.4 | 2 8 | 21 1 | 2 16 | 20 7 | 2 13 | 20 7.8 | 2 15.6 | 21 0.2 |
| 10 30 | 2 21 | 21 9 | 2 21.2 | 22 0.4 | 2 16 | 21 10.7 | 2 19 | 21 11 | 2 12.3 | 21 8.4 |
| 11 30 | 2 11 | 22 2.4 | 2 4.6 | 22 4 | 2 7 | 22 3.5 | 2 8 | 22 4.9 | 2 7.3 | 21 7.3 |
| | | | 3° Dec. S. | | 6° Dec. S. | | 9° Dec. S. | | 12° Dec. S. | |
| 0 30 | | | 1 54.4 | 22 7 | 1 55 | 22 8.8 | 1 56 | 22 8.3 | 1 47.5 | 22 8.6 |
| 1 30 | | | 1 41.4 | 22 6 | 1 43 | 22 11 | 1 36 | 22 9 | 1 39.2 | 22 9.2 |
| 2 30 | | | 1 18 | 22 3.4 | 1 23.4 | 22 11 | 1 25.2 | 22 9.8 | 1 21.5 | 23 0 |
| 3 30 | | | 1 16 | 22 6.5 | 1 9.5 | 22 7 | 1 10.6 | 22 11.9 | 1 6.3 | 22 5.5 |
| 4 30 | | | 0 59 | 21 6.9 | 1 1 | 22 9 | 0 57.6 | 21 9.4 | 1 0 | 21 8.1 |
| 5 30 | | | 0 53.7 | 21 1.7 | 0 56 | 21 5 | 0 55 | 21 0.2 | 0 47.5 | 21 0.3 |
| 6 30 | | | 0 56 | 20 3.7 | 0 56.5 | 20 0.8 | 0 56.3 | 20 4.3 | 0 55 | 20 3.5 |
| 7 30 | | | 1 31.5 | 19 6.7 | 1 29 | 19 8.5 | 1 22.7 | 20 1.8 | 1 19.3 | 19 7 |
| 8 30 | | | 2 0.3 | 20 5.8 | 1 59.3 | 20 4.2 | 1 52 | 20 1.2 | 1 49 | 20 3.8 |
| 9 30 | | | 2 15 | 21 1 | 2 15 | 20 10 | 2 9 | 21 0 | 2 9.4 | 20 9.6 |
| 10 30 | | | 2 15 | 21 11 | 2 11 | 21 10 | 2 14 | 21 9.3 | 2 15 | 21 7 |
| 11 30 | | | 2 8.6 | 22 6.5 | 2 8 | 22 5 | 2 5 | 22 4 | 2 7 | 22 10 |

| Time of Moon's Transit. | 15° Dec. N. | | 18° Dec. N. | | 21° Dec. N. | | 24° Dec. N. | | 27° Dec. N. | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Interval. | Height. | Interval. | Height. | Interval. | Height. | Interval. | Height. | Interval. | Height. |
| h m | h m | ft. in. | h m | ft. in. | h m | ft. in. | h m | ft. in. | h m | ft. in. |
| 0 30 | 1 51.9 | 22 5.5 | 1 49.5 | 22 7 | 1 50 | 22 7 | 1 45 | 22 6.4 | 1 42 | 22 5.4 |
| 1 30 | 1 39.5 | 22 9 | 1 33.4 | 22 11 | 1 28 | 22 9 | 1 31.6 | 22 8.9 | 1 24.7 | 22 8.2 |
| 2 30 | 1 22 | 22 7.2 | 1 21 | 23 0 | 1 17 | 22 6 | 1 17 | 22 7.1 | 1 11 | 23 0 |
| 3 30 | 1 6.6 | 22 5.4 | 1 3.3 | 22 5.3 | 0 51.4 | 22 4.9 | 0 54 | 22 2.4 | 0 52.3 | 22 3.2 |
| 4 30 | 0 50 | 21 8 | 0 53.2 | 21 10.5 | 0 42 | 21 9.9 | 0 40 | 21 5.4 | 0 37.5 | 21 7.8 |
| 5 30 | 0 40 | 21 3.2 | 0 36.7 | 20 10 | 0 40.3 | 20 6 | 0 29.1 | 20 10 | 0 30.7 | 20 10.6 |
| 6 30 | 0 42.5 | 20 2.3 | 0 38.3 | 19 9.6 | 0 32 | 19 10.4 | 0 24.2 | 19 3.6 | 0 26 | 19 3 |
| 7 30 | 1 10 | 19 3.3 | 0 57.5 | 19 1.5 | 0 56.6 | 18 5.4 | 0 41 | 19 0.6 | 0 29 | 18 10.2 |
| 8 30 | 1 43 | 20 0.4 | 1 38.6 | 19 5.6 | 1 31.4 | 19 3 | 1 13 | 19 4 | 1 13.7 | 18 10.6 |
| 9 30 | 2 3.3 | 20 6 | 2 0.5 | 20 3 | 1 50.3 | 20 1 | 1 58.7 | 19 10 | 2 2.6 | 20 2.6 |
| 10 30 | 2 13.5 | 21 6.8 | 2 3.7 | 21 2.5 | 2 5 | 21 1.2 | 2 4.6 | 21 3.3 | 2 1 | 20 10 |
| 11 30 | 2 9 | 21 9.6 | 2 3 | 22 1 | 1 59.4 | 22 3 | 1 50 | 21 7.2 | 1 52.5 | 21 8.6 |
| | 15° Dec. S. | | 18° Dec. S. | | 21° Dec. S. | | 24° Dec. S. | | 27° Dec. S. | |
| 0 30 | 1 48 | 23 0 | 1 48 | 22 7.5 | 1 42 | 22 4.7 | 1 42 | 22 6.4 | 1 37 | 22 4.8 |
| 1 30 | 1 37 | 22 10.9 | 1 29 | 22 10.3 | 1 28.4 | 23 0.3 | 1 26.6 | 22 7.9 | 1 21 | 21 11.4 |
| 2 30 | 1 17.8 | 22 10.4 | 1 13 | 22 11 | 1 10.6 | 22 7 | 1 9 | 22 6 | 1 8.6 | 22 5.6 |
| 3 30 | 1 2.6 | 22 6.4 | 0 59 | 22 8.4 | 0 56.9 | 22 6.6 | 0 51.7 | 21 10.8 | 0 47.7 | 22 1.2 |
| 4 30 | 0 49.5 | 21 10.7 | 0 49.2 | 21 10.2 | 0 47.3 | 21 8 | 0 40 | 21 8.4 | 0 28.3 | 21 7.4 |
| 5 30 | 0 45 | 20 7.8 | 0 42.3 | 21 0.1 | 0 35.5 | 20 8.5 | 0 43 | 20 9.2 | 0 26 | 20 5.5 |
| 6 30 | 0 47 | 19 10.6 | 0 37 | 20 0 | 0 38 | 20 3 | 0 29 | 19 8 | 0 20.5 | 19 7.5 |
| 7 30 | 1 10 | 19 3.1 | 0 58 | 19 6.3 | 1 1.7 | 19 5.1 | 0 49.5 | 19 4.3 | 0 33 | 18 6.4 |
| 8 30 | 1 40 | 20 0.2 | 1 39 | 19 8.8 | 1 37 | 19 7 | 1 17 | 19 4.1 | 1 22 | 19 3 |
| 9 30 | 2 2 | 20 8.4 | 1 58 | 20 7 | 2 3.6 | 20 4.6 | 1 57.7 | 20 1.7 | 1 44 | 20 0 |
| 10 30 | 2 7.6 | 21 3.9 | 2 2 | 21 4.1 | 2 5 | 21 1.9 | 2 6.4 | 21 2.5 | 1 53 | 20 9 |
| 11 30 | 2 1.3 | 22 2.7 | 1 58 | 22 1.8 | 1 51.4 | 22 3.6 | 1 48.5 | 21 10.4 | 1 51 | 21 7.2 |

## Table XI. (Interpolated from Table IX.)

Showing the Interval between the Moon's Transit and the Time of High Water at the London Docks, for every three degrees of her Declination, North or South.

| Moon's Transit. | 0 | 3° Dec. | 6° Dec. | 9° Dec. | 12° Dec. | 15° Dec. | 18° Dec. | 21° Dec. | 24° Dec. | 27° Dec. | Mean. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m | h m |
| 0 0 | 2 5 | 2 1.5 | 2 3.6 | 2 1.8 | 1 59 | 1 59 | 1 54.3 | 1 50.7 | 1 46.3 | 1 46.6 | 1 58.8 |
| 0 30 | 1 58.9 | 1 56.4 | 1 58.9 | 1 57 | 1 48.2 | 1 49.9 | 1 48.2 | 1 46 | 1 43.5 | 1 39.5 | 1 50.6 |
| 1 0 | 1 49.8 | 1 50.5 | 1 51.7 | 1 46.8 | 1 44 | 1 44.2 | 1 39.7 | 1 37.2 | 1 36.2 | 1 31 | 1 42.8 |
| 1 30 | 1 40 | 1 42.7 | 1 42 | 1 37.1 | 1 38.6 | 1 38.3 | 1 31.2 | 1 28.2 | 1 29 | 1 22.8 | 1 35 |
| 2 0 | 1 32 | 1 34 | 1 32.7 | 1 30.6 | 1 29.6 | 1 29.3 | 1 24 | 1 21.3 | 1 21.2 | 1 16.3 | 1 27.1 |
| 2 30 | 1 24.5 | 1 26 | 1 23.5 | 1 25 | 1 20.4 | 1 19.9 | 1 17 | 1 14.4 | 1 13 | 1 9.8 | 1 19.3 |
| 3 0 | 1 19.3 | 1 20 | 1 17 | 1 17.3 | 1 14.5 | 1 12.3 | 1 9 | 1 4.2 | 1 3.3 | 1 0 | 1 11.7 |
| 3 30 | 1 13.7 | 1 14 | 1 12.3 | 1 10 | 1 7.5 | 1 4.6 | 1 1.1 | 0 54.1 | 0 52.9 | 0 50 | 1 4 |
| 4 0 | 1 5.3 | 1 6.7 | 1 6 | 1 3.5 | 1 3 | 0 56.8 | 0 56.1 | 0 49.3 | 0 46 | 0 40.5 | 0 57.3 |
| 4 30 | 0 59 | 0 59.5 | 0 59.5 | 0 59 | 0 59 | 0 49.7 | 0 51.2 | 0 44.6 | 0 40 | 0 32.9 | 0 51.4 |
| 5 0 | 0 58 | 0 57.2 | 0 56.8 | 0 56.8 | 0 53.8 | 0 45 | 0 45.3 | 0 41.2 | 0 36 | 0 29.5 | 0 48 |
| 5 30 | 0 59.8 | 0 56.4 | 0 55 | 0 55.5 | 0 48.7 | 0 42.5 | 0 39.5 | 0 37.9 | 0 32.8 | 0 28.3 | 0 45.6 |
| 6 0 | 1 2 | 0 58.4 | 0 54.8 | 0 55.5 | 0 48 | 0 42.8 | 0 37 | 0 36.3 | 0 29.5 | 0 25.5 | 0 44.9 |
| 6 30 | 1 5.1 | 1 3.2 | 0 56.3 | 0 57.7 | 0 52 | 0 44.7 | 0 37.6 | 0 35 | 0 26.6 | 0 23.3 | 0 46.1 |
| 7 0 | 1 12.3 | 1 10.2 | 1 4 | 1 8.6 | 1 1.5 | 0 56 | 0 45.8 | 0 44 | 0 33 | 0 24.6 | 0 54 |
| 7 30 | 1 21.3 | 1 21.4 | 1 19.1 | 1 23.3 | 1 15.1 | 1 10 | 0 57.8 | 0 59.1 | 0 45.2 | 0 31 | 1 6.3 |
| 8 0 | 1 36 | 1 40 | 1 38 | 1 38 | 1 30 | 1 26.4 | 1 16 | 1 16 | 0 59 | 0 52 | 1 23.1 |
| 8 30 | 1 55 | 1 54.6 | 1 54.6 | 1 51.5 | 1 47.5 | 1 41.5 | 1 38.8 | 1 34.2 | 1 15 | 1 17.8 | 1 41 |
| 9 0 | 2 9 | 2 4.5 | 2 8 | 2 2.7 | 2 3 | 1 53.5 | 1 51 | 1 48 | 1 38 | 1 39 | 1 55.7 |
| 9 30 | 2 18.2 | 2 11.5 | 2 15.5 | 2 11 | 2 12.5 | 2 2.6 | 1 59.2 | 1 58 | 1 58.2 | 1 53.3 | 2 6 |
| 10 0 | 2 21.3 | 2 16 | 2 16 | 2 15.5 | 2 14.2 | 2 8 | 2 2.1 | 2 3.3 | 2 6.8 | 1 57 | 2 9.8 |
| 10 30 | 2 21 | 2 18.2 | 2 13.5 | 2 16.5 | 2 13.6 | 2 10.5 | 2 2.8 | 2 5 | 2 5.5 | 1 57 | 2 10.4 |
| 11 0 | 2 17.6 | 2 15 | 2 11 | 2 12.8 | 2 11.3 | 2 9.5 | 2 2.5 | 2 2.5 | 1 59.5 | 1 55 | 2 7.7 |
| 11 30 | 2 11 | 2 6.6 | 2 7.5 | 2 6.5 | 2 7.1 | 2 6.4 | 2 0.5 | 1 55.4 | 1 49.2 | 1 51.7 | 2 2.2 |

## Table XII. (Interpolated from Table IX.)

Showing the Height of High Water at the London Docks for every three degrees of the Moon's Declination, North or South.

| Moon's Transit. | 0 | 3° Dec. | 6° Dec. | 9° Dec. | 12° Dec. | 15° Dec. | 18 Dec. | 21° Dec. | 24° Dec. | 27° Dec. | Mean. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| h m | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. |
| 0 0 | 22.36 | 22.53 | 22.52 | 22.48 | 22.50 | 22.40 | 22.36 | 22.38 | 22.12 | 22.04 | 22.37 |
| 0 30 | 22.70 | 22.62 | 22.69 | 22.58 | 22.77 | 22.73 | 22.60 | 22.48 | 22.53 | 22.43 | 22.61 |
| 1 0 | 22.95 | 22.62 | 22.83 | 22.71 | 22.70 | 22.79 | 22.74 | 22.75 | 22.65 | 22.37 | 22.71 |
| 1 30 | 23.10 | 22.63 | 22.92 | 22.84 | 22.72 | 22.83 | 22.88 | 22.80 | 22.70 | 22.31 | 22.78 |
| 2 0 | 23.20 | 22.55 | 22.90 | 22.95 | 22.83 | 22.79 | 22.95 | 22.77 | 22.65 | 22.55 | 22.81 |
| 2 30 | 23.11 | 22.47 | 22.80 | 22.97 | 22.94 | 22.73 | 22.96 | 22.54 | 22.54 | 22.73 | 22.78 |
| 3 0 | 22.86 | 22.49 | 22.62 | 22.86 | 22.80 | 22.63 | 22.80 | 22.50 | 22.31 | 22.50 | 22.64 |
| 3 30 | 22.61 | 22.52 | 22.39 | 22.70 | 22.55 | 22.49 | 22.56 | 22.48 | 22.05 | 22.19 | 22.45 |
| 4 0 | 22.24 | 22.09 | 22.64 | 22.30 | 22.10 | 22.18 | 22.22 | 22.13 | 21.78 | 21.92 | 22.16 |
| 4 30 | 21.89 | 21.67 | 21.88 | 21.82 | 21.62 | 21.78 | 21.86 | 21.75 | 21.49 | 21.64 | 21.74 |
| 5 0 | 21.49 | 21.20 | 21.65 | 21.35 | 21.27 | 21.37 | 21.40 | 20.20 | 21.16 | 21.20 | 21.23 |
| 5 30 | 21.10 | 20.79 | 21.41 | 20.92 | 20.92 | 20.96 | 20.92 | 20.60 | 20.80 | 20.67 | 20.91 |
| 6 0 | 20.50 | 20.52 | 20.88 | 20.55 | 20.53 | 20.50 | 20.40 | 20.32 | 20.18 | 20.00 | 20.44 |
| 6 30 | 19.84 | 20.30 | 20.12 | 20.20 | 20.15 | 20.04 | 19.90 | 20.04 | 19.48 | 19.44 | 19.95 |
| 7 0 | 19.75 | 19.54 | 19.87 | 20.12 | 19.82 | 19.60 | 19.56 | 19.50 | 19.32 | 19.03 | 19.61 |
| 7 30 | 19.92 | 19.78 | 19.72 | 20.04 | 19.55 | 19.27 | 19.32 | 18.94 | 19.21 | 18.70 | 19.44 |
| 8 0 | 19.91 | 20.03 | 19.87 | 20.00 | 19.90 | 19.60 | 19.36 | 19.08 | 19.23 | 18.82 | 19.58 |
| 8 30 | 19.92 | 20.41 | 20.13 | 19.95 | 20.44 | 20.06 | 19.60 | 19.42 | 19.33 | 19.06 | 19.83 |
| 9 0 | 20.53 | 20.75 | 20.42 | 20.25 | 20.68 | 20.35 | 20.00 | 19.82 | 19.60 | 19.59 | 19.20 |
| 9 30 | 21.20 | 21.08 | 20.71 | 20.82 | 20.91 | 20.60 | 20.42 | 20.23 | 19.98 | 20.11 | 20.61 |
| 10 0 | 21.50 | 21.50 | 21.29 | 21.36 | 21.28 | 20.98 | 20.86 | 20.66 | 20.55 | 20.48 | 21.05 |
| 10 30 | 21.75 | 21.97 | 21.87 | 21.85 | 21.64 | 21.45 | 21.28 | 21.14 | 21.24 | 20.79 | 21.50 |
| 11 0 | 22.00 | 22.26 | 22.16 | 22.16 | 21.98 | 21.90 | 21.70 | 21.71 | 21.55 | 21.24 | 21.97 |
| 11 30 | 22.22 | 22.44 | 22.36 | 22.38 | 22.22 | 22.02 | 22.12 | 22.28 | 21.72 | 21.66 | 22.14 |

## Table XIII.

Showing the Time of High Water at the London Docks corresponding to the Mean Time of the Moon's superior and inferior Transits, and also the Heights of High Water, in the Month of June, from 1090 Observations, made between the 1st of January 1808 and 31st of December 1826.

| Superior. | | | | Inferior. | | | |
|---|---|---|---|---|---|---|---|
| Moon's Transit. | Time of High Water. | Height of Tide. | No. of Obs. | Moon's Transit. | Time of High Water. | Height of Tide. | No. of Obs. |
| h m | h m | ft. in. | | h m | h m | ft. in. | |
| 0 16 | 2 3.2 | 22 5 | 20 | 0 15 | 2 0 | 22 3.4 | 19 |
| 0 44 | 2 22.5 | 22 5 | 20 | 0 45 | 2 24.2 | 22 5 | 20 |
| 1 11.8 | 2 45 | 22 8 | 22 | 1 14.9 | 2 44.5 | 22 7.2 | 19 |
| 1 41.9 | 3 10.2 | 22 7.8 | 23 | 1 45 | 3 9.8 | 22 6.5 | 23 |
| 2 15.2 | 3 30.5 | 22 6.5 | 20 | 2 17.7 | 3 33.6 | 22 7.6 | 22 |
| 2 46.8 | 3 55.6 | 22 7.2 | 26 | 2 44 | 3 53.1 | 22 6.4 | 16 |
| 3 17.2 | 4 22.6 | 22 6.4 | 17 | 3 13 | 4 14.3 | 22 4.8 | 27 |
| 3 44.9 | 4 40.3 | 22 3.6 | 29 | 3 46.7 | 4 43 | 21 10.1 | 28 |
| 4 16.9 | 5 11.3 | 21 10.9 | 19 | 4 19.7 | 5 7.1 | 21 11.1 | 26 |
| 4 44.6 | 5 35.9 | 21 4.8 | 29 | 4 46.2 | 5 37.4 | 21 4.4 | 21 |
| 5 17.6 | 6 10.2 | 21 3.3 | 23 | 5 14.4 | 6 5.2 | 21 1.7 | 27 |
| 5 44.5 | 6 37.7 | 20 7.8 | 26 | 5 47.4 | 6 42.7 | 20 8.1 | 22 |
| 6 15.7 | 7 16.6 | 20 2.7 | 25 | 6 15.4 | 7 10.2 | 20 5.6 | 22 |
| 6 45.5 | 7 51.5 | 20 0.8 | 26 | 6 45.3 | 7 50.7 | 20 2.2 | 29 |
| 7 16 | 8 35.2 | 19 8.7 | 24 | 7 16.5 | 8 37 | 19 9.8 | 22 |
| 7 45.3 | 9 16 | 19 9.2 | 25 | 7 46.5 | 9 21.5 | 19 11 | 23 |
| 8 15.6 | 9 58.5 | 20 1.9 | 24 | 8 15.3 | 9 57.2 | 19 10.4 | 23 |
| 8 45.4 | 10 44.3 | 20 2.6 | 23 | 8 45.3 | 10 39.6 | 20 7.4 | 25 |
| 9 16 | 11 21 | 20 9.2 | 25 | 9 17.4 | 11 19.5 | 20 8.3 | 22 |
| 9 46.1 | 11 56 | 21 1.1 | 20 | 9 45.5 | 11 50.8 | 20 11.9 | 19 |
| 10 16.2 | 12 21.7 | 21 4.3 | 23 | 10 14 | 12 22.7 | 21 2.4 | 22 |
| 10 43 | 12 52.7 | 21 8.1 | 20 | 10 48.9 | 12 46.8 | 21 7.5 | 22 |
| 11 16.1 | 13 11.1 | 22 1.4 | 22 | 11 15.5 | 13 18.3 | 21 9.6 | 18 |
| 11 45.6 | 13 40.7 | 22 0.5 | 20 | 11 45.3 | 13 38.2 | 22 2.6 | 22 |

## Table XIV. (Interpolated from Table XIII.)

Showing the Interval between the Moon's Transit and the Times of High Water, and the Height of High Water in the month of June; the Interval and Heights corresponding to the Moon's superior and inferior Transits being distinguished.

| | Superior. | | Inferior. | | Difference. | |
|---|---|---|---|---|---|---|
| Moon's Transit. | High Water. | Height. | High Water. | Height. | | |
| h m | h m | Feet. | h m | Feet. | m | |
| 0 0 | 1 52 | 22.21 | 1 49 | 22.35 | +3 | −.04 |
| 0 30 | 1 43.8 | 22.42 | 1 42.1 | 22.35 | +1.7 | +.07 |
| 1 0 | 1 35.5 | 22.56 | 1 34.4 | 22.51 | +1.1 | +.05 |
| 1 30 | 1 30.3 | 22.80 | 1 27.2 | 22.57 | +3.1 | +.23 |
| 2 0 | 1 21.3 | 22.59 | 1 20.9 | 22.58 | +0.4 | +.01 |
| 2 30 | 1 12.2 | 22.56 | 1 12.8 | 22.58 | −0.6 | +.02 |
| 3 0 | 1 7.4 | 22.56 | 1 4.6 | 22.46 | +2.8 | +.1 |
| 3 30 | 1 1.4 | 22.39 | 0 58.7 | 22.12 | +2.7 | +.27 |
| 4 0 | 0 55 | 22.11 | 0 55.3 | 21.90 | −0.3 | +.21 |
| 4 30 | 0 52.9 | 21.66 | 0 52.7 | 21.64 | +0.2 | +.02 |
| 5 0 | 0 51.9 | 21.34 | 0 51 | 21.26 | +0.9 | +.08 |
| 5 30 | 0 52.9 | 20.99 | 0 53 | 20.92 | −0.1 | +.07 |
| 6 0 | 0 56.9 | 20.45 | 0 55 | 20.58 | +1.9 | −.13 |
| 6 30 | 1 3.3 | 20.15 | 1 0.1 | 20.33 | +3.2 | −.18 |
| 7 0 | 1 15.5 | 19.90 | 1 12.2 | 20.01 | +3.3 | −.11 |
| 7 30 | 1 29.9 | 19.74 | 1 27.2 | 19.87 | +2.7 | −.13 |
| 8 0 | 1 36.8 | 19.96 | 1 38.5 | 19.89 | −1.7 | +.07 |
| 8 30 | 1 50.8 | 20.19 | 1 48.1 | 20.25 | +2.7 | −.06 |
| 9 0 | 2 2 | 20.47 | 1 57.9 | 20.64 | +4.1 | −.17 |
| 9 30 | 2 7.5 | 20.92 | 2 3.7 | 20.84 | +3.8 | +.08 |
| 10 0 | 2 7.8 | 21.22 | 2 7 | 21.10 | +0.8 | +.12 |
| 10 30 | 2 7.7 | 21.53 | 2 3.8 | 21.39 | +3.9 | +.14 |
| 11 0 | 2 2.3 | 21.91 | 2 0.1 | 21.70 | +2.2 | +.21 |
| 11 30 | 1 55 | 22.08 | 1 57.8 | 22.00 | −2.8 | +.08 |

## Table XV.

Showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) in different Months of the Year.

| Moon's Transit. | Jan. | Feb. | March. | April. | May. | June. | July. | Aug. | Sept. | Oct. | Nov. | Dec. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| h m | m | m | m | m | m | m | m | m | m | m | m | m |
| 0 0 | − 5 | + 2 | + 4 | + 7 | − 2 | − 7 | − 4 | ..... | + 9 | + 7 | − 3 | − 7 |
| 0 30 | − 5 | + 1 | + 7 | + 4 | − 2 | − 8 | − 5 | + 6 | + 8 | + 5 | − 6 | − 6 |
| 1 0 | − 3 | + 3 | + 6 | + 4 | − 3 | − 7 | − 1 | + 8 | + 8 | + 3 | − 8 | − 7 |
| 1 30 | − 2 | + 3 | + 5 | + 3 | − 4 | − 7 | − 1 | + 8 | + 8 | + 2 | − 9 | − 6 |
| 2 0 | − 1 | + 3 | + 8 | + 3 | − 6 | − 6 | − 1 | +11 | + 8 | + 2 | −10 | − 7 |
| 2 30 | + 2 | + 6 | + 7 | + 2 | − 8 | − 6 | 0 | +11 | + 7 | 0 | −10 | − 9 |
| 3 0 | + 2 | + 6 | + 5 | + 1 | − 9 | − 5 | + 2 | +10 | + 7 | − 4 | −11 | − 8 |
| 3 30 | + 5 | + 8 | + 4 | − 4 | − 8 | − 4 | + 6 | +11 | + 5 | − 7 | −13 | − 6 |
| 4 0 | +10 | + 7 | + 2 | − 6 | − 8 | − 1 | +10 | +12 | + 4 | − 9 | −14 | − 3 |
| 4 30 | +11 | + 6 | + 1 | − 8 | − 9 | + 2 | +12 | +10 | 0 | −11 | −11 | − 2 |
| 5 0 | +13 | + 3 | − 4 | −12 | − 9 | + 5 | +15 | + 9 | − 3 | −15 | −11 | + 1 |
| 5 30 | +16 | ..... | − 9 | −14 | − 7 | + 9 | +16 | + 8 | − 6 | −17 | − 8 | + 7 |
| 6 0 | +16 | − 2 | −12 | −14 | − 4 | +14 | +18 | + 5 | −10 | −18 | − 4 | +13 |
| 6 30 | +12 | − 4 | −15 | −14 | + 1 | +18 | +18 | + 1 | −14 | −15 | − 2 | +18 |
| 7 0 | +13 | −11 | −22 | −12 | + 7 | +21 | +14 | − 5 | −19 | −14 | + 8 | +16 |
| 7 30 | + 5 | −20 | −29 | − 8 | +10 | +22 | +12 | −10 | −19 | − 1 | +16 | +15 |
| 8 0 | 0 | −27 | −26 | − 4 | +14 | +16 | + 5 | −11 | −13 | + 6 | +18 | +14 |
| 8 30 | − 4 | −25 | −14 | + 2 | +17 | +10 | − 1 | −10 | − 5 | + 7 | +16 | +12 |
| 9 0 | − 7 | −23 | − 9 | + 5 | +13 | + 3 | − 8 | −11 | − 1 | +13 | +15 | + 6 |
| 9 30 | − 7 | −19 | − 4 | + 8 | +11 | 0 | −10 | − 9 | + 1 | +15 | +13 | + 1 |
| 10 0 | − 8 | − 8 | + 2 | + 6 | + 7 | − 3 | −11 | − 5 | + 4 | +14 | + 5 | − 4 |
| 10 30 | − 7 | − 7 | + 5 | + 6 | + 4 | − 4 | − 8 | − 4 | + 7 | +10 | + 4 | − 3 |
| 11 0 | − 6 | − 1 | + 3 | + 6 | + 4 | − 7 | − 7 | − 2 | + 7 | + 8 | + 3 | − 3 |
| 11 30 | − 6 | ..... | + 4 | + 7 | + 2 | − 6 | − 5 | − 1 | + 7 | + 7 | ..... | − 3 |

## Table XVI.

Showing the Difference in the Height of High Water and the Mean Height (Column B. Table V.) in different Months of the Year.

| Moon's Transit. | Jan. | Feb. | March. | April. | May. | June. | July. | Aug. | Sept. | Oct. | Nov. | Dec. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| h m | m | m | m | m | m | m | m | m | m | m | m | m |
| 0 0 | −.16 | −.23 | −.02 | +.21 | +.29 | −.16 | −.51 | −.33 | +.10 | +.51 | +.22 | +.03 |
| 0 30 | −.56 | +.10 | +.22 | +.27 | +.12 | −.20 | −.29 | −.28 | +.11 | +.37 | +.45 | −.26 |
| 1 0 | −.36 | −.02 | +.13 | +.28 | +.15 | −.19 | −.24 | −.16 | +.15 | +.40 | +.32 | −.51 |
| 1 30 | −.29 | −.01 | +.19 | +.15 | +.15 | −.19 | −.14 | −.09 | +.10 | +.35 | +.24 | −.37 |
| 2 0 | −.04 | −.09 | −.02 | +.12 | +.15 | −.26 | −.09 | −.15 | +.23 | +.18 | +.12 | −.08 |
| 2 30 | −.10 | +.25 | 0 | +.21 | −.02 | −.18 | −.06 | +.05 | +.26 | −.12 | −.18 | −.11 |
| 3 0 | +.08 | +.13 | −.16 | +.04 | −.12 | −.25 | −.02 | +.10 | +.26 | −.01 | +.20 | −.24 |
| 3 30 | +.02 | ..... | +.09 | −.11 | −.16 | −.25 | +.09 | +.10 | +.34 | +.23 | −.03 | −.28 |
| 4 0 | +.04 | +.12 | +.19 | −.12 | −.21 | −.05 | +.27 | −.02 | +.20 | −.16 | −.16 | −.13 |
| 4 30 | +.17 | +.07 | +.06 | −.08 | −.23 | −.03 | +.33 | −.06 | +.12 | −.11 | −.16 | −.08 |
| 5 0 | +.31 | +.06 | +.12 | −.06 | −.34 | −.07 | +.33 | +.32 | +.04 | −.42 | −.38 | +.03 |
| 5 30 | +.17 | +.30 | +.23 | −.29 | −.51 | −.02 | +.40 | +.31 | −.03 | −.23 | −.30 | +.07 |
| 6 0 | +.52 | +.09 | +.09 | −.29 | −.34 | +.21 | +.54 | +.35 | −.07 | −.65 | −.48 | +.10 |
| 6 30 | +.18 | −.34 | +.07 | −.41 | −.06 | +.41 | +.62 | +.21 | −.14 | −.09 | −.40 | +.21 |
| 7 0 | +.27 | −.19 | −.26 | −.55 | +.23 | +.42 | +.63 | +.26 | −.36 | −.26 | −.38 | +.26 |
| 7 30 | +.24 | −.72 | −.17 | −.47 | +.34 | +.39 | +.59 | +.16 | −.29 | −.35 | +.17 | +.19 |
| 8 0 | −.10 | −.34 | −.55 | −.36 | +.55 | +.52 | +.60 | −.07 | −.43 | −.28 | +.20 | +.24 |
| 8 30 | −.16 | −.74 | −.44 | −.32 | +.52 | +.58 | +.37 | −.28 | −.41 | +.13 | −.05 | +.50 |
| 9 0 | −.17 | −.78 | −.67 | −.12 | +.58 | +.52 | +.14 | −.38 | −.30 | +.10 | +.55 | +.58 |
| 9 30 | −.30 | −.68 | −.40 | +.16 | +.58 | +.34 | −.19 | −.50 | −.31 | +.62 | +.19 | +.57 |
| 10 0 | −.41 | −.72 | −.34 | +.33 | +.61 | +.26 | −.25 | −.44 | −.11 | +.46 | +.25 | +.37 |
| 10 30 | −.29 | −.45 | −.29 | +.32 | +.65 | +.08 | −.44 | −.54 | −.04 | +.65 | +.38 | +.06 |
| 11 0 | −.40 | −.40 | −.21 | +.49 | +.50 | +.09 | −.48 | −.49 | +.09 | +.34 | +.60 | −.13 |
| 11 30 | −.44 | −.89 | +.06 | +.53 | +.53 | +.01 | −.39 | −.25 | +.14 | +.37 | +.48 | −.13 |

## TABLE XVII.

Showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) for every Minute of the Moon's Horizontal Parallax.

| Moon's Transit. | H. P. 54′. | H. P. 55′. | H. P. 56′. | H. P. 57′. | H. P. 58′. | H. P. 59′. | H. P. 60′. | H. P. 61′. |
|---|---|---|---|---|---|---|---|---|
| h m | m | m | m | m | m | m | m | m |
| 0 0 | +12 | + 9 | + 4 | ..... | − 3 | − 4 | −13 | −14 |
| 0 30 | +12 | + 9 | + 2 | + 2 | − 3 | − 5 | − 9 | −11 |
| 1 0 | +10 | + 8 | + 3 | + 5 | − 1 | − 4 | − 9 | −11 |
| 1 30 | + 8 | + 5 | + 3 | + 5 | − 1 | − 3 | −10 | −11 |
| 2 0 | + 8 | + 6 | + 2 | + 3 | + 1 | − 1 | − 8 | − 9 |
| 2 30 | + 7 | + 5 | + 1 | + 1 | + 2 | − 2 | − 6 | − 8 |
| 3 0 | + 6 | + 4 | + 2 | ..... | + 2 | − 2 | − 6 | |
| 3 30 | + 6 | + 4 | + 3 | + 1 | + 3 | − 2 | − 5 | |
| 4 0 | + 4 | + 3 | + 2 | − 1 | + 2 | − 2 | − 6 | |
| 4 30 | ..... | + 1 | + 3 | − 1 | − 1 | − 2 | − 8 | |
| 5 0 | + 1 | + 1 | + 3 | + 1 | 0 | − 1 | | |
| 5 30 | + 1 | 0 | − 1 | + 2 | − 1 | − 1 | | |
| 6 0 | + 1 | + 1 | − 3 | + 1 | − 2 | − 2 | | |
| 6 30 | + 2 | + 4 | − 3 | − 1 | − 3 | − 3 | | |
| 7 0 | + 4 | + 2 | − 3 | − 2 | − 5 | − 4 | | |
| 7 30 | + 9 | − 2 | − 2 | − 4 | − 7 | − 7 | − 7 | |
| 8 0 | +16 | 0 | 0 | − 3 | − 8 | − 6 | −11 | |
| 8 30 | +21 | + 8 | + 3 | + 1 | − 7 | − 6 | −12 | |
| 9 0 | +19 | + 9 | + 4 | + 1 | − 9 | −10 | −16 | |
| 9 30 | +17 | +11 | + 7 | + 1 | − 8 | −11 | −18 | |
| 10 0 | +16 | +12 | + 8 | 0 | − 5 | −12 | −17 | |
| 10 30 | +15 | +13 | + 8 | − 1 | − 2 | −10 | −14 | −17 |
| 11 0 | +13 | +12 | + 7 | − 2 | − 2 | − 8 | −14 | −18 |
| 11 30 | +13 | +10 | + 6 | − 1 | − 2 | − 4 | −16 | −16 |

## TABLE XVIII.

Showing the Difference in the Height of High Water, and the Mean Height (Column B. Table V.) for every Minute of the Moon's Horizontal Parallax.

| Moon's Transit. | H. P. 54′. | H. P. 55′. | H. P. 56′. | H. P. 57′. | H. P. 58′. | H. P. 59′. | H. P. 60′. | H. P. 61′. |
|---|---|---|---|---|---|---|---|---|
| h m | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. |
| 0 0 | −0.52 | −.33 | −.30 | +.10 | +.06 | +.23 | + .33 | +.53 |
| 0 30 | − .50 | −.28 | −.20 | +.08 | +.17 | +.41 | + .36 | +.53 |
| 1 0 | − .58 | −.44 | −.09 | −.07 | +.26 | +.24 | + .31 | +.61 |
| 1 30 | − .55 | −.46 | +.03 | −.07 | +.23 | +.23 | + .41 | +.77 |
| 2 0 | − .57 | −.52 | −.15 | −.05 | +.12 | +.24 | + .50 | +.75 |
| 2 30 | − .54 | −.45 | −.28 | +.10 | +.08 | +.34 | + .69 | +.79 |
| 3 0 | − .68 | −.37 | −.25 | +.06 | +.05 | +.42 | + .83 | |
| 3 30 | − .63 | −.32 | −.13 | +.08 | +.13 | +.56 | +1.00 | |
| 4 0 | − .57 | −.28 | −.13 | +.14 | +.30 | +.68 | + .98 | |
| 4 30 | − .49 | −.29 | −.13 | +.23 | +.67 | +.76 | + .83 | |
| 5 0 | − .50 | −.40 | −.30 | +.25 | +.53 | +.80 | | |
| 5 30 | − .47 | −.40 | −.37 | +.37 | +.50 | +.86 | | |
| 6 0 | − .45 | −.28 | −.19 | +.27 | +.50 | +.77 | | |
| 6 30 | − .54 | −.21 | −.09 | +.05 | +.40 | +.62 | + .43 | |
| 7 0 | − .64 | −.29 | −.11 | +.05 | +.31 | +.54 | + .53 | |
| 7 30 | − .75 | −.48 | −.28 | +.01 | +.15 | +.33 | + .58 | |
| 8 0 | − .68 | −.30 | −.25 | +.05 | +.02 | −.08 | + .54 | |
| 8 30 | − .54 | −.27 | −.26 | +.01 | −.20 | +.42 | + .44 | |
| 9 0 | − .26 | −.23 | −.21 | −.11 | −.23 | −.76 | + .39 | |
| 9 30 | + .03 | −.24 | −.15 | −.26 | −.09 | +.30 | + .45 | +.87 |
| 10 0 | − .09 | −.30 | −.11 | −.14 | +.14 | +.36 | + .49 | +.69 |
| 10 30 | − .20 | −.37 | −.07 | +.03 | +.32 | +.36 | + .48 | +.55 |
| 11 0 | − .31 | −.31 | −.14 | +.21 | +.26 | +.26 | + .51 | +.61 |
| 11 30 | − .43 | −.22 | −.25 | +.27 | +.13 | +.14 | + .46 | +.68 |

## Table XIX.

Showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) for every three degrees of the Moon's Declination.

| Moon's Transit. | 0 | 3° Dec. | 6° Dec. | 9° Dec. | 12° Dec. | 15° Dec. | 18° Dec. | 21° Dec. | 24° Dec. | 27° Dec. |
|---|---|---|---|---|---|---|---|---|---|---|
| h m | m | m | m | m | m | m | m | m | m | m |
| 0 0 | + 8 | + 5 | + 7 | + 5 | + 2 | + 2 | − 3 | − 6 | −11 | −10 |
| 0 30 | + 9 | + 6 | + 9 | + 7 | − 2 | ..... | − 2 | − 4 | − 7 | −11 |
| 1 0 | + 8 | + 8 | +10 | + 5 | + 2 | + 2 | − 3 | − 5 | − 6 | −11 |
| 1 30 | + 5 | + 8 | + 7 | + 2 | + 4 | + 3 | − 4 | − 7 | − 6 | −12 |
| 2 0 | + 6 | + 8 | + 6 | + 4 | + 3 | + 3 | − 2 | − 5 | − 5 | −10 |
| 2 30 | + 6 | + 8 | + 5 | + 7 | + 2 | + 2 | − 1 | − 4 | − 5 | − 9 |
| 3 0 | + 9 | + 9 | + 6 | + 7 | + 4 | + 2 | − 2 | − 7 | − 7 | −11 |
| 3 30 | +11 | +11 | + 9 | + 7 | + 5 | + 2 | − 2 | − 9 | −10 | −13 |
| 4 0 | + 9 | +10 | +10 | + 7 | + 7 | + 1 | 0 | − 7 | −10 | −16 |
| 4 30 | + 8 | + 8 | + 8 | + 8 | + 8 | − 1 | 0 | − 6 | −11 | −18 |
| 5 0 | +13 | +12 | +11 | +12 | + 9 | 0 | 0 | − 4 | − 9 | −16 |
| 5 30 | +17 | +14 | +12 | +13 | + 6 | 0 | − 3 | − 5 | −10 | −14 |
| 6 0 | +20 | +16 | +13 | +13 | + 6 | + 1 | − 5 | − 7 | −13 | −17 |
| 6 30 | +21 | +19 | +12 | +13 | + 8 | ..... | − 7 | − 9 | −18 | −21 |
| 7 0 | +21 | +19 | +12 | +17 | +10 | + 4 | − 6 | − 8 | −19 | −27 |
| 7 30 | +16 | +16 | +14 | +18 | +10 | + 5 | − 8 | − 6 | −20 | −34 |
| 8 0 | +14 | +18 | +16 | +16 | + 8 | + 4 | − 6 | − 6 | −23 | −30 |
| 8 30 | +16 | +15 | +15 | +12 | + 8 | + 2 | − 1 | − 5 | −24 | −22 |
| 9 0 | +13 | + 9 | +12 | + 7 | + 7 | − 2 | − 5 | − 8 | −18 | −17 |
| 9 30 | +13 | + 7 | +11 | + 6 | + 7 | − 2 | − 6 | − 7 | − 7 | −12 |
| 10 0 | +11 | + 6 | + 6 | + 6 | + 4 | − 2 | − 8 | − 7 | − 5 | −13 |
| 10 30 | +11 | + 8 | + 3 | + 6 | + 3 | 0 | − 7 | − 5 | − 5 | −13 |
| 11 0 | + 9 | + 7 | + 3 | + 4 | + 3 | + 1 | − 6 | − 6 | − 9 | −13 |
| 11 30 | + 8 | + 3 | + 4 | + 3 | + 4 | + 3 | − 3 | − 8 | −14 | −12 |

## Table XX.

Showing the Difference in the Height of High Water, and the Mean Height (Column B. Table V.) for every three degrees of the Moon's Declination.

| Moon's Transit. | 0 | 3° Dec. | 6° Dec. | 9° Dec. | 12° Dec. | 15° Dec. | 18° Dec. | 21° Dec. | 24° Dec. | 27° Dec. |
|---|---|---|---|---|---|---|---|---|---|---|
| h m | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. | Feet. |
| 0 0 | −.10 | +.07 | +.06 | +.02 | +.04 | −.06 | −.10 | −.08 | −.34 | −.42 |
| 0 30 | +.09 | −.01 | +.06 | −.05 | +.14 | +.10 | −.03 | −.15 | −.10 | −.20 |
| 1 0 | +.23 | −.10 | +.11 | −.01 | −.02 | +.07 | +.02 | +.03 | −.07 | −.35 |
| 1 30 | +.34 | −.13 | +.16 | +.08 | −.04 | +.07 | +.12 | +.13 | −.06 | −.45 |
| 2 0 | +.40 | −.25 | +.10 | +.15 | +.03 | −.01 | +.15 | −.03 | −.15 | −.35 |
| 2 30 | +.39 | −.25 | +.08 | +.25 | +.22 | +.01 | +.24 | −.18 | −.18 | +.01 |
| 3 0 | +.27 | −.10 | +.03 | +.27 | +.21 | +.04 | +.21 | −.09 | −.28 | −.09 |
| 3 30 | +.22 | +.13 | ..... | +.31 | +.16 | +.10 | +.17 | +.01 | −.34 | −.20 |
| 4 0 | +.14 | −.01 | +.54 | +.20 | ..... | +.08 | +.12 | +.03 | −.32 | −.18 |
| 4 30 | +.17 | −.05 | +.16 | +.10 | −.10 | +.06 | +.14 | −.08 | −.13 | −.08 |
| 5 0 | +.21 | −.08 | +.37 | +.07 | −.01 | +.09 | +.12 | −.28 | −.12 | −.08 |
| 5 30 | +.22 | −.09 | +.53 | +.04 | +.04 | +.08 | +.04 | −.05 | −.08 | −.21 |
| 6 0 | +.13 | +.15 | +.51 | +.18 | +.16 | +.13 | +.03 | +.11 | −.19 | −.37 |
| 6 30 | −.09 | +.37 | +.19 | +.27 | +.22 | +.11 | −.03 | −.06 | −.45 | −.49 |
| 7 0 | +.19 | −.02 | +.31 | +.56 | +.26 | +.04 | 0 | −.46 | −.24 | −.53 |
| 7 30 | +.52 | +.38 | +.32 | +.64 | +.15 | −.13 | −.06 | −.35 | −.19 | −.70 |
| 8 0 | +.48 | +.60 | +.44 | +.57 | +.47 | +.17 | −.07 | −.28 | −.20 | −.61 |
| 8 30 | +.22 | +.71 | +.43 | +.25 | +.74 | +.36 | −.10 | −.28 | −.37 | −.64 |
| 9 0 | +.43 | +.65 | +.32 | +.15 | +.58 | +.25 | −.10 | −.28 | −.50 | −.41 |
| 9 30 | +.66 | +.54 | +.17 | +.28 | +.37 | +.06 | −.12 | −.31 | −.56 | −.43 |
| 10 0 | +.58 | +.58 | +.37 | +.44 | +.36 | +.06 | −.06 | −.26 | −.37 | −.44 |
| 10 30 | +.32 | +.54 | +.44 | +.42 | +.21 | +.02 | −.15 | −.29 | −.19 | −.64 |
| 11 0 | +.15 | +.41 | +.31 | +.31 | +.13 | +.05 | −.15 | −.14 | −.30 | −.61 |
| 11 30 | +.12 | +.34 | +.26 | +.28 | +.12 | −.08 | +.02 | +.18 | −.38 | −.44 |

## Table XXI.

Showing the Influence of the Direction of the Wind on the Time and Height of High Water.

| Direction of Wind. | Number of Tides observed. | Sum of Times of High Water. | | Mean Error. | Sum of Heights of High Water. | | Mean Error. |
|---|---|---|---|---|---|---|---|
| | | Calculated. | Observed. | | Calculated. | Observed. | |
| | | h m | h m | m | ft. in. | ft. in. | inches. |
| N | 38 | 202 43 | 201 43 | −2 | 812 4 | 813 4 | ..... |
| NNE | 53 | 271 20 | 272 45 | +2 | 1150 2 | 1148 4 | ..... |
| NE | 81 | 359 50 | 362 25 | +2 | 1766 11 | 1772 9 | +1 |
| ENE | 43 | 230 54 | 229 55 | −1 | 938 3 | 951 4 | +4 |
| E | 38 | 126 8 | 127 0 | +1 | 829 7 | 834 9 | +2 |
| ESE | 16 | 96 47 | 95 25 | −5 | 348 5 | 344 5 | −3 |
| SE | 23 | 125 0 | 125 5 | ..... | 485 6 | 485 8 | ..... |
| SSE | 18 | 117 34 | 117 55 | +1 | 372 8 | 369 3 | −2 |
| S | 29 | 148 58 | 149 30 | +1 | 608 8 | 603 6 | −2 |
| SSW | 36 | 262 25 | 264 40 | +4 | 734 3 | 736 1 | ..... |
| SW | 148 | 877 29 | 874 0 | −1 | 3129 2 | 3123 2 | ..... |
| WSW | 55 | 286 52 | 283 35 | −4 | 1189 8 | 1179 8 | −2 |
| W | 36 | 221 57 | 220 0 | −3 | 771 9 | 771 1 | ..... |
| WNW | 10 | 70 40 | 69 55 | −4 | 210 7 | 209 10 | ..... |
| NW | 37 | 189 51 | 188 35 | −2 | 810 4 | 810 11 | ..... |
| NNW | 36 | 197 41 | 197 10 | −1 | 755 2 | 754 3 | ..... |

## XXII. *On the extensive Atmosphere of Mars.* By Sir James South, *F.R.S.* *Communicated by His Royal Highness the President.*

Read June 16, 1831.

THAT several of the planets as well as that which we inhabit are surrounded by atmosphere, astronomical observations have long since established; the extent, however, to which in particular planets such atmospheres are diffused, is as yet not satisfactorily determined. The former rests principally upon phenomena observed on the planets' discs, whilst the latter derives its support chiefly from those detected at or near their respective limbs. Every night, nay almost every hour, may give us indication of the one, whilst years are sometimes necessary, as in the case of planets unattended by satellites, to help us to the other; thus the hypothesis of the *extensive* atmosphere of Mars derives its origin from the observations of Cassini and Roëmer, and has stood more than a century and a half without refutation or support.

The observations to which I allude formed part of a series undertaken for the determination of the parallax of Mars, and are recorded in the Mémoires de l'Académie des Sciences. Cassini's were made at Briare and at La Charité sur Loire; whilst Roëmer's was obtained at the Royal Observatory of Paris.

Of the Briare observation Cassini says, "Le premier Octobre 1672 à $2^h$ $45^m$ du matin à Briare, Mars vû par une lunette de trois pieds, sembloit toucher par son bord septentrional, la ligne droite tirée par la première et par la seconde de l'eau d'Aquarius marquée ψ, d'où il n'étoit plus éloigné que de six minutes. Cette étoile paroissoit si diminuée et si affoiblie de lumière, qu'on ne la pouvoit plus distinguer ni à la vue simple, ni par une lunette un peu plus foible."—Mém. de l'Acad. tome vii. p. 357.

The La Charité and Paris observations entitled "Eclipse de la Moyenne ψ dans l'eau d'Aquarius" are thus narrated: "Quoique le ciel fut alors assez beau de part et d'autre, et que l'on vit Mars pendant un assez long espace de temps,

on ne vit point l'étoile moyenne ♄, qui devoit être cachée par son disque. Le diamètre de Mars étoit alors de 25 secondes.—Mém. de l'Acad. tom. vii. p. 358.

" Les nuages qui survinrent ne permirent pas d'en voir la sortie ; et l'on ne sçait pas même si on l'auroit pû voir immédiatement, car trois quarts d'heure après le ciel s'étant découvert à Paris, M. Roëmer la chercha attentivement autour de Mars, et il ne la trouva qu'après l'attention de deux minutes, quand elle étoit déjà éloignée bu bord oriental de Mars de deux tiers de son diamètre. C'étoit alors $11^h$ $15^m$, et le parallèle de l'étoile coupoit le diamètre de Mars en raison de 2 à 3. Il commença de la voir sans difficulté, quand elle étoit éloignée de Mars de trois quarts de son diamètre."—Mém. de l'Acad. tom. vii. p. 359.

Hence we learn that a star of the fifth magnitude at the distance of six minutes from the planet Mars became invisible to Cassini ; and that after occultation by the planet, the same star could not be detected by Roëmer, till the planet's limb had receded from it, almost seventeen seconds of a degree. Experience, however, has long shown us that stars of the same magnitude are visible even when in actual contact with the moon's enlightened limb ; to what cause then is the invisibility of the star when in the vicinity of Mars referable ? Cassini attributed it to the atmosphere of Mars ; and although it seems difficult to imagine one of such *enormous extent* as the Briare observation would require, still, as any other hypothesis would involve us in greater difficulty, I shall adopt it, and shall present it, as also the comments which precede it, in Cassini's own words : " Cette difficulté de voir cette étoile de la cinquième grandeur très proche de Mars est considérable, d'autant qu'il n'y a point de difficulté à voir des étoiles de la même grandeur jusqu'au bord de la lune. Ce qui pourroit fair juger que Mars est environné de quelque atmosphère."—Mém. de l'Acad. tom. vii. p. 359.

Admitting, then, that an *extensively* diffused atmosphere of Mars is indicated by the observations above quoted, let us see if modern observations can confirm it.

On the 27th October 1783, Sir William Herschel, with a new 20-feet Reflector of 18.7 inches aperture, saw a star of the 13th or 14th magnitude at a distance of two minutes and fifty-six seconds from the planet, " not otherwise

affected by the approach of Mars, than what the brightness of its superior light might account for."—Vide Phil. Trans. vol. lxxiv. p. 272.

On the 19th of February 1822, in Blackman-street, a star of the 9th or 10th magnitude was for several hours seen in the field of the 5-feet Equatorial with the planet Mars. At a distance of one minute and forty-three seconds of a degree from the planet, (which took place at $11^h\ 15^m$ sidereal time,) its splendour suffered no sensible diminution.

On the following night a star 42 Leonis, of the 6th magnitude, was in the field of the same instrument with Mars, and the planet's progress towards the star was observed micrometrically for several hours; nor did the star suffer any loss of its brilliancy as its distance from the planet diminished.

Fatigued by previous watchings, at about two in the morning I retired to rest; but thinking it probable that the star would undergo occultation, accompanied by my brother Mr. Henry South I returned to the instrument about 4 o'clock, and found Mars about half his own diameter from the star. The planet had about twenty-four degrees of altitude; its limb was at times well defined and steady, at other times extremely unsteady; the star was comparatively steady*, could be kept tolerably well upon the micrometer-wire, and was of a beautiful blue colour. At $15^h\ 3^m\ 23^s.3$ sidereal time, it was seen admirably defined, and was distant from the limb of the planet a diameter of one of the micrometer-wires, equal nearly to one second of a degree; from which time till $15^h\ 3^m\ 53^s$ the planet's limb was so extremely unsteady and ill defined, that the precise moment of occultation could not be obtained.

After emersion, at $15^h\ 20^m\ 38^s.3$ the star was seen when it was about $1\frac{1}{2}$ diameter of the wire, or one second and a half from the limb; it was almost indigo blue; and the contrast between it and the planet, which was of a deep red, was exquisitely beautiful. By reference to the double star 48 $i$ Cancri, some idea of it may be entertained, if we regard the larger of its stars as Mars, and the smaller as 42 Leonis. At the time of observation the planet had passed his opposition only forty-seven hours, and his apparent diameter as measured with the micrometer, was sixteen seconds and six tenths.

* This steadiness of one sidereal object, as contrasted with the extreme unsteadiness of another, seen under similar circumstances of atmosphere and altitude, has *long* been familiar to me, and is a phenomenon to which, on some future occasion, I shall probably invite the Society's attention.

The phenomena were also witnessed by Mr. Henry South, and his observations strictly accorded with my own. To render our results as independent as possible of optical misrepresentation, his instrument was my Gregorian reflector, by Watson, of six inches aperture and thirty inches focus; the figure of its metal was exquisitely perfect, whilst twelve years of constant use, had not occasioned the slightest tarnish of its almost colourless surface.

Accustomed as the first business of the night, to point a telescope to such of the principal planets as are above the horizon, on the 17th of March of the present year I had the satisfaction of seeing in the field of the 5-feet Equatorial, with the planet Mars, several stars, some of them minute, but one of the 5th and another of the 6th magnitude. Convinced that the planet would pass close by, or perhaps occult the larger, I took its place, and found it to be 37 Tauri. The covering under which the dome for my large equatorial was being built, unfortunately rendered the 5-feet equatorial useless, when the planet had approached within forty seconds of the star. The observations therefore were continued with the 12-feet Achromatic of $7\frac{3}{4}$ inches aperture, and also with the 42-inch of $2\frac{3}{4}$ aperture, till the star was one diameter and a half of the planet from his nearest limb; when, fearing lest the trees to the north-west of my grounds, might intercept the planet from my view, at the instant of nearest appulse or occultation, the 8-feet Achromatic of six inches aperture previously placed on the top of the house, was recurred to. The star, from being a full diameter distant from the planet when first observed with this telescope, was watched most unremittingly till the planet, having been in contact with it, had receded from it a quantity equal to its own semi-diameter. The star suffered not the least change of colour, nor the least diminution of its lustre, except what of the latter might fairly be attributed to the splendour of the planet; its rays were certainly in contact with the planet's limb, but only at their circumference; at times the planet and star were very steady, at other times far otherwise; but at no period was there such contrast between the steadiness of the star and the unsteadiness of the planet, as occurred at the occultation of 42 Leonis. As the star's distance from the planet diminished, the former seemed to undergo not the slightest alteration; and when in actual contact, both the star and the planet were red, but the planet had the deeper tint. The night was remark-

ably fine, and although Mars had not more than seven or eight degrees of altitude, a power of 320 was used with advantage.

The diameter of Mars, as taken with the 5-feet Equatorial, was about ten seconds. The observations were commenced at about seven hours sidereal time, whilst the nearest appulse was perhaps at about eleven hours; they were not made for determining the place of Mars, but for noticing any phenomena which the star might exhibit.

The facts being now before us, the inferences may be comprised in a few lines.

Sir W. Herschel's observation of the 27th of October 1783, and mine of the small star on the 19th of February 1822, are at variance with Cassini's observation, but impugn not the accuracy of Roemer's; whilst my observations of 42 Leonis and of 37 Tauri, being apparently subversive of the observations both of Cassini and of Roëmer, point out the "*extensive* atmosphere of Mars" as a subject meriting further investigation.

Such are all the observations relative to the *extensive* atmosphere of Mars which my observatory can furnish. One of these, viz. that of the 17th of last March, demands further consideration, lest, having served to invalidate one hypothesis, it might be brought forward to support another; namely, that "the red colour of the planet Mars is dependent upon the physical properties of his atmosphere."* Moreover, it seems inconsistent with a previous observation.

The star 37 Tauri had "nearly the colour of Mars" whilst in contact with the planet; whereas, the star 42 Leonis was "beautifully blue" previously and subsequently to occultation by the planet. The facts are different;—are they reconcileable? The following then are extracts from the Observatory Journal. On the night following the observation of 37 Tauri, namely, the 18th of March, "the five-feet Equatorial was placed upon Mars, in order to compare its colour with that of 37 Tauri; the star and planet were still in the field together, though nearly at opposite points of its circumference. "I can have no hesitation in saying, that the star is red, but not of so deep a tint as the planet."

"Mars being placed out of the field, I requested an attendant (accustomed to use a telescope) to look at the star and to tell me its colour; entirely un-

* Vide Brewster's Encyclopædia, vol. ii. pages 636 & 637 (article Mars).

acquainted with the bearing of the question, and ignorant of the colour assigned to it by me, he replied, 'Certainly a light red.' Mars was now brought into the field with the star, and being asked what colour he now considered the star to have, he answered, 'Certainly red, but not so deep a red as Mars.' Looking at it again, he said, 'Mars is the darker, but there is not a great deal of difference.'"

"I now applied several other eye-pieces, magnifying from 70 to 548 times, and with all of them I felt convinced that the star was red, but not so deep a red as Mars."

"Whilst the colour of 37 Tauri was fresh in my recollection," I placed the Equatorial upon 42 Leonis, when it had nearly the same altitude as that star; I instantly pronounced the star 42 Leonis to be blue—light blue. The attendant was now requested to look again at 37 Tauri, and to retain its colour in his mind as much as possible: 42 Leonis was next brought into the field; he said, "This star is certainly not red at all; I do not know what colour it is, unless a light blue." Alternating the examination of the one star with that of the other several times, and with various powers, he at last said, "It is certainly blue, and the first is certainly red."

The comparisons were repeated with the 12-feet Achromatic of 7¾ inches aperture, and the inferences drawn from them were the same; and, if just, the observations of 37 Tauri and 42 Leonis are perfectly reconcileable. Hypothesis, therefore, is not needed, to explain under similar circumstances with regard to Mars, the "red" colour of the one star, or the "blue" colour of the other.

## XXIII. *On the Friction and Resistance of Fluids.* *By* GEORGE RENNIE, *Esq.* *V.P.R.S.*

Read June 16, 1831.

WHEN on a former occasion I communicated the results of a series of experiments on the Friction and Resistance of the Surfaces of Solids (Philosophical Transactions for 1828), I stated that they formed part only of a series of experiments on the nature of friction generally. My object at first was to trace the relation subsisting between the retardation produced by the surfaces of solids in motion when in contact with each other and with fluids; but finding that the subject connected with either of these branches was sufficiently extensive, I deemed it necessary to postpone the second part of the inquiry to a future occasion. Those experiments, however, established some important facts. They showed that (within the limits of abrasion) friction was the same for all solids, and that it was neither affected by surface nor velocity. Subsequent experiments upon rolling bodies of great weight and magnitude, when the resistance was reduced $\frac{1}{1000}$th part of the mass, and the surfaces in the ratio of 13 to 1, have corroborated the affinity of resistance between rolling and sliding bodies. Thus in connecting and continuing the isolated experiments of COULOMB and VINCE, and assigning values to the abrasive resistances of most of the most useful solids, a considerable advance has been made in the science.

The subject of the present paper, however, involves difficulties of a more complicated kind. The theory of solids as deduced from the laws of mechanics, and independent of experiment, may be applied to any system of bodies; but the theory of fluids, in which the form and the disposition of the particles, or the laws of their action, are unknown, must necessarily be founded on experiment; and even with this aid, which can only be obtained through the intervention of a solid, our knowledge of the true properties of fluids must be vague

and uncertain. Accordingly we find that the subject of fluids attracted the attention of some of the most distinguished mathematicians and philosophers of Europe for the last two centuries; that is, from the year 1628, when CASTELLI first published his Treatise on the Measure of running Water, down to the hydraulic investigations of EYTELWEIN and YOUNG. Between these periods, Italy, France, Germany and England, added their contributions to the science. But it is to the Italians principally that we owe the foundation of it, in their numerous investigations and controversies on the rivers of Italy; hence the writings of CASTELLI, VIVIANI, ZENDRINI, MANFREDI, POLINI, FRISI, GULIELMINI, LECHI, MICHELLOTTI, and of many others *.

Each of them has endeavoured to establish a theory applicable to rivers and torrents, but in general with indifferent success. The science again received fresh accessions from the more valuable investigations of BOSSUT, DUBUAT, VENTURI, FÜNCK, BRUNNING, BIDONE, COULOMB, PRONY, EYTELWEIN and GIRARD; and among our own countrymen, of M'CLAURIN, VINCE, MATTHEW YOUNG, Dr. JURIN, Professor ROBINSON, and the late Dr. THOMAS YOUNG. Sir ISAAC NEWTON had already demonstrated, in his celebrated propositions 51, 52, and 53, of the Principia, (in the case of a cylinder in motion immersed in a fluid,) that the resistance arising from the want of a perfect lubricity in fluids is (cæteris paribus) proportional to the velocity with which the parts of a fluid separated from each other; and that, if a solid cylinder of infinite length revolves with a uniform motion round a fixed axis, in a uniform and infinite fluid, the periodical times of the parts of the fluid thus put in motion will be proportional to their distances from the axis. This theory (although conformable to experiment) was objected to by BERNOULLI and D'ALEMBERT, on the ground that Sir ISAAC NEWTON had not taken into consideration the centrifugal force or friction arising from the pressure of the concentric rings or filaments round the cylinder, the fluid being supposed in a state of permanence, and the friction of the rings equal throughout.

PITOT (1728), in his experiments on the water-works at Marly and Versailles, was the first to demonstrate that with equal velocities, and in the ratio of the volume of water, the friction of water in pipes was in the inverse ratio of their

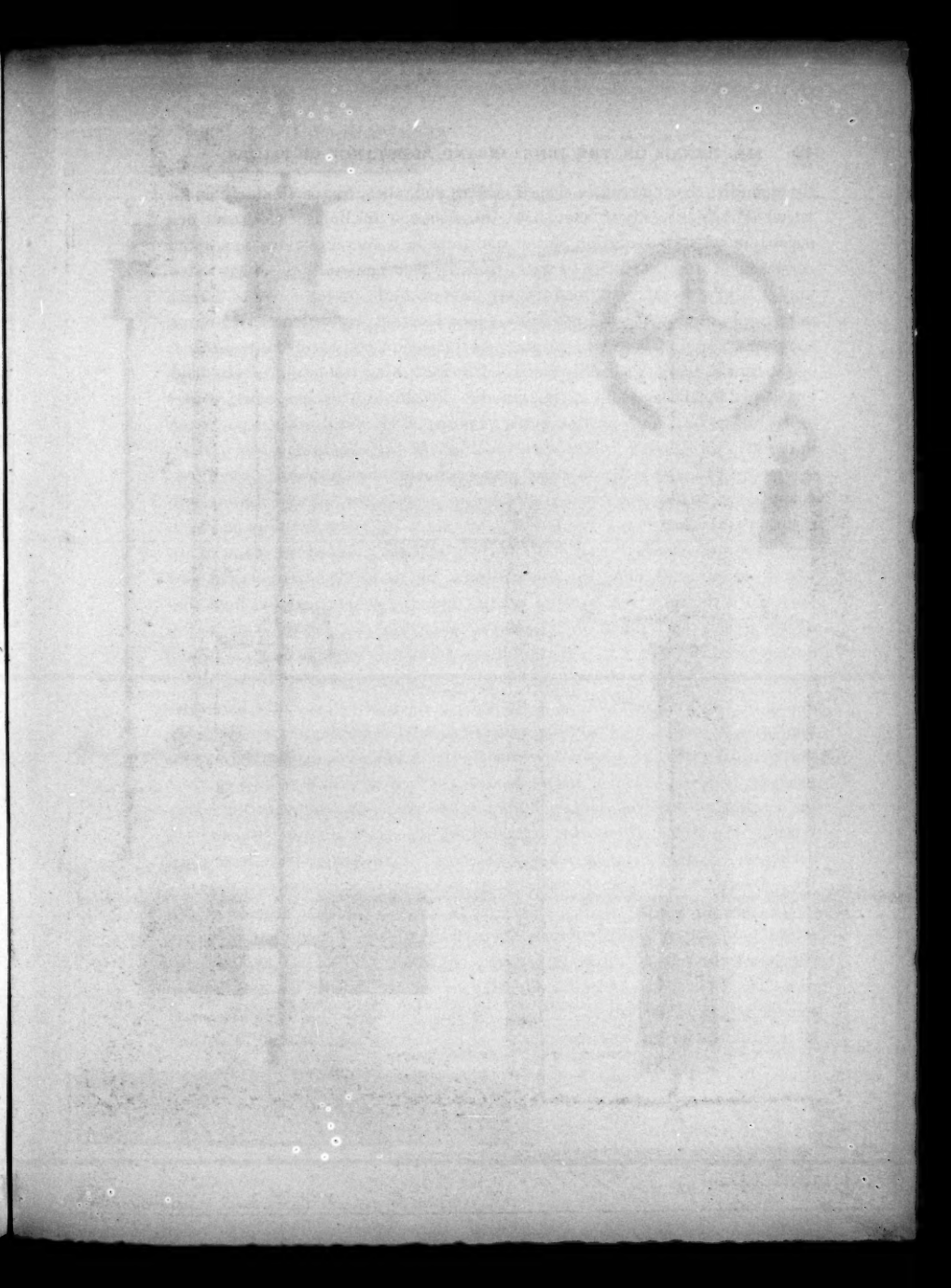
* Raccotta d' Autori che trattano del Moto dell' Acque.

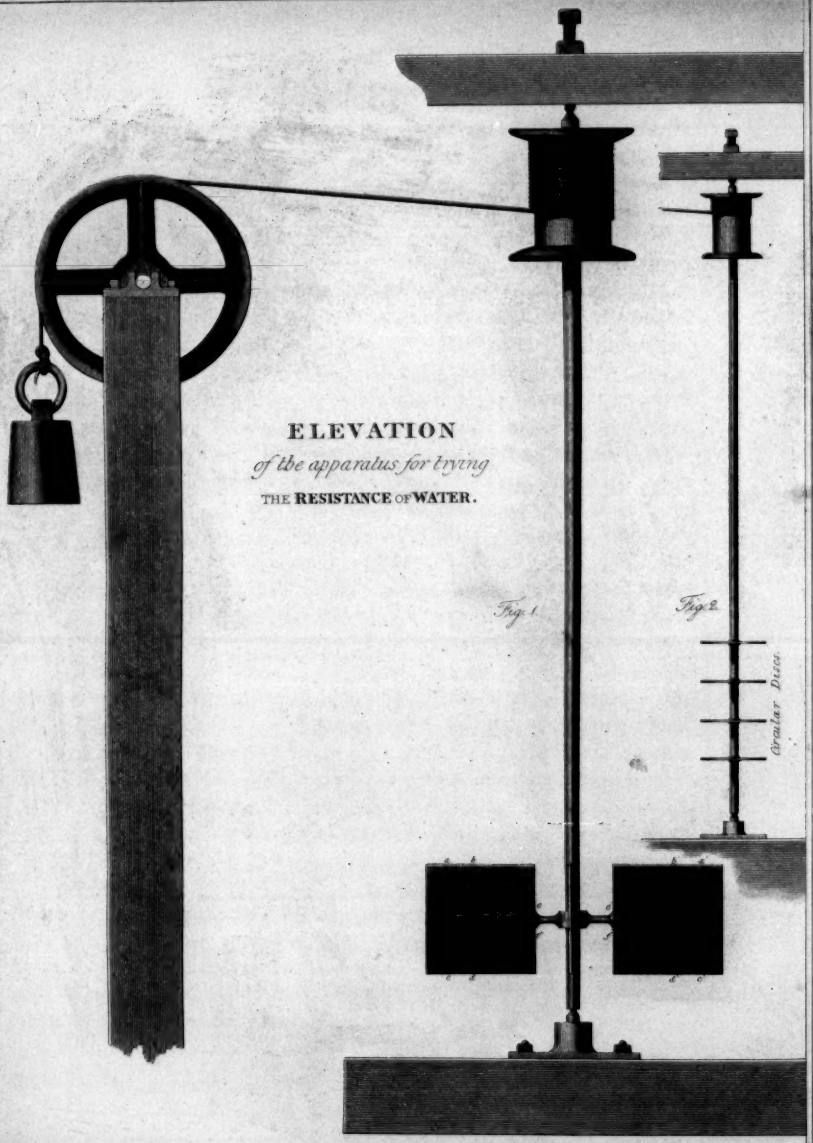
diameters; and COUPLET (1733), MARIOTTE, and DEPARCIEUX, estimated the difference between the real and calculated expenditures of glass tubes and pipes.

CHEZY (in 1771 and 1786) was the first engineer who endeavoured to establish the relation subsisting between the inclination of an aqueduct and the transverse section of the volume of water it ought to carry,—on the supposition that the accelerating force, due to the inclination of the bed of the conduit, is counterbalanced by the resistances of the channel in the ratio of the surface, and increasing in proportion to the square of the velocity. What CHEZY had remarked was concluded by BOSSUT, who cleared the investigation of most of its difficulties, and demonstrated it to be in accordance with theory. He found that small orifices discharged less water in proportion than great ones, on account of friction; that the vena contracta, and consequent expenditure, diminished with the height of the reservoir: he pointed out the law by which the discharge diminishes according to the inclination and number of bends in a pipe, and the influence of friction in retarding the velocity of waters moving in canals and pipes, in which he made the square of the velocity to be in the inverse ratio of the length of the pipe: he determined the co-efficients by experiment, and thus obtained a formula expressive of the conditions of the uniform motion of water in open canals. The greater part of these hypotheses may be said to have been removed by the more extensive researches of DUBUAT. His great hydraulic work, published in 1779 and 1786, contains a series of the most valuable observations, whose results accord very nearly with the new formula of the motion of water in pipes and open conduits; and his experiments, with pipes inclined in various angles from the 40,000th part of a right angle to 90 degrees, and in channels which varied from a line and a half in diameter to areas of seven or eight square toises, seem to comprehend every case of inclination; so that by collecting a prodigious number of facts, both with compressible and incompressible fluids, he obtained a general expression for all cases relative to the friction and cohesion of fluids: but a logarithmic function which he introduces in it, by a sort of approximation, gives it a character of uncertainty, which restrains its use, and shows the necessity of fresh researches. VENTURI, in 1798, "Sur la Communication latérale du Mouvements dans les Fluides," repeated and added many new facts to the experiments of BOSSUT, on

the expenditure of differently shaped orifices and tubes, but particularly on the lateral communication of motion by the cohesion of fluids. Coulomb first approximated to the solution of the question, by a very ingenious apparatus, consisting of discs of different sizes, fixed by their centres to the lower extremity of a brass wire, and made to oscillate in fluids by the force of torsion only; he concluded that the resistance was a function, composed of two terms, one proportional to the first, the other to the second powers of the resistance: again, that it was not sensibly increased by increasing the height of the fluid, but simply by the cohesion of the particles of the fluid which presented greater or less resistance, in proportion to the viscidity of the fluid, oil being to water in the ratio of 17.5 to 1. But whatever might be the conclusions of Coulomb, it is obvious that both the size and construction of his apparatus were ill calculated to produce results whereon to found a satisfactory theory; and accordingly both Messrs. Prony and Girard, in expressing their formulæ of resistance, have not admitted that of Coulomb, but have adopted the mean of the best of experiments made by other authors: but as these formulæ give only the mean velocity, which is much greater than the velocity (of the fluid contiguous to the pipe) which ought alone to enter into the expression of the retarding force, it follows, that the coefficients deduced from the mean of all the experiments adopted by these gentlemen, have a value greatly inferior to the motion of the fluid contiguous to the side of the pipe or conduit. To ascertain correctly the value of this kind of resistance, M. Girard (vide les Mémoires des Sçavans etrangers for 1815), undertook a prodigious number of experiments on tubes of different diameters and length, from which he deduced that the retardation is as the velocity simply. The effects of temperature are very remarkable; if the velocity be expressed by 10, when the temperature is 0° centigrade thermometer, the velocity will be 42°, or increased four times when the temperature is 85°: these values must be deemed approximations only.

The contributions of British philosophers towards the improvement of this science have been, unfortunately, scanty; for, with the exception of Sir Isaac Newton (who led the way), Dr. Jurin, Dr. Matthew Young, Dr. Desaguliers, Dr. Vince, Mr. Smeaton, Mr. Banks, and the late Dr. Thomas Young, (see the paper of the latter gentleman in the Philosophical Transactions, and his commentaries on Eytelwein's experiments,) we can scarcely find any





# ELEVATION

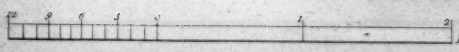
*of the apparatus for trying*

THE **RESISTANCE** OF **WATER**.

*Fig. 1.*

*Fig. 2.*

*Circular Discs.*



experiments on the subject*: whatever has been effected by our engineers or scientific men, has either been withheld from the public, or consigned to obscurity; and though we have tracts of marshes and fen land, consisting of many thousand acres, the dissertations on the mode of draining and carrying off their superfluous waters are confined to local pamphlets and reports, of comparatively minor interest to the science of hydraulics.

From the foregoing short but imperfect history, it is obvious that much has been done towards perfecting this science. It is however certain, that much yet remains to be accomplished; and although we are deeply indebted to both the French and English philosophers for their extensive investigations on the laws of capillary attraction, the descents of globes in fluids, and the adhesion of fluids to metal discs, the phenomena of fluidity, and the laws which govern the motion and equilibrium of their particles, must yet remain a problem purely geometrical; and as we possess no tangible means of approximating to the solution of the problem, but through the intervention of a solid, we must content ourselves, in like manner, with the imperfect formulæ deduced from experiments made on a small scale on the friction and adhesion of water in pipes and conduits, until we can ascertain more correctly the causes of the retardations of rivers as they occur in nature.

In the consideration of this question, therefore, I propose to examine, first, the retardations of the surfaces of solids moving in fluids at rest; secondly, the retardations of fluids over solids; and, thirdly, the direct resistance of solids revolving in fluids at rest.

To illustrate the first case, I caused an apparatus to be constructed, of which the annexed Plate XI. is a representation; it consists simply of a cylinder of wood ten inches and three quarters in diameter, and twenty-four inches long, and divided into eight sections of three inches in each, and fixed upon a spindle of iron about four feet in length, and one inch and a quarter thick. The apparatus was accurately turned and polished. Upon the upper part of the spindle, a small cylinder or pulley, six inches in diameter was fixed, and a fine flexible silken cord, communicating with the weight, was wound;

* The experiments of the Society for the Improvement of Naval Architecture, in the years 1793, 1794 1795, 1796, 1797, 1798, relate principally to the resistances of solids moving through fluids.

the apparatus was then fixed in an iron frame, and the frame let into a groove in two upright posts, driven into the bed of the river Thames.

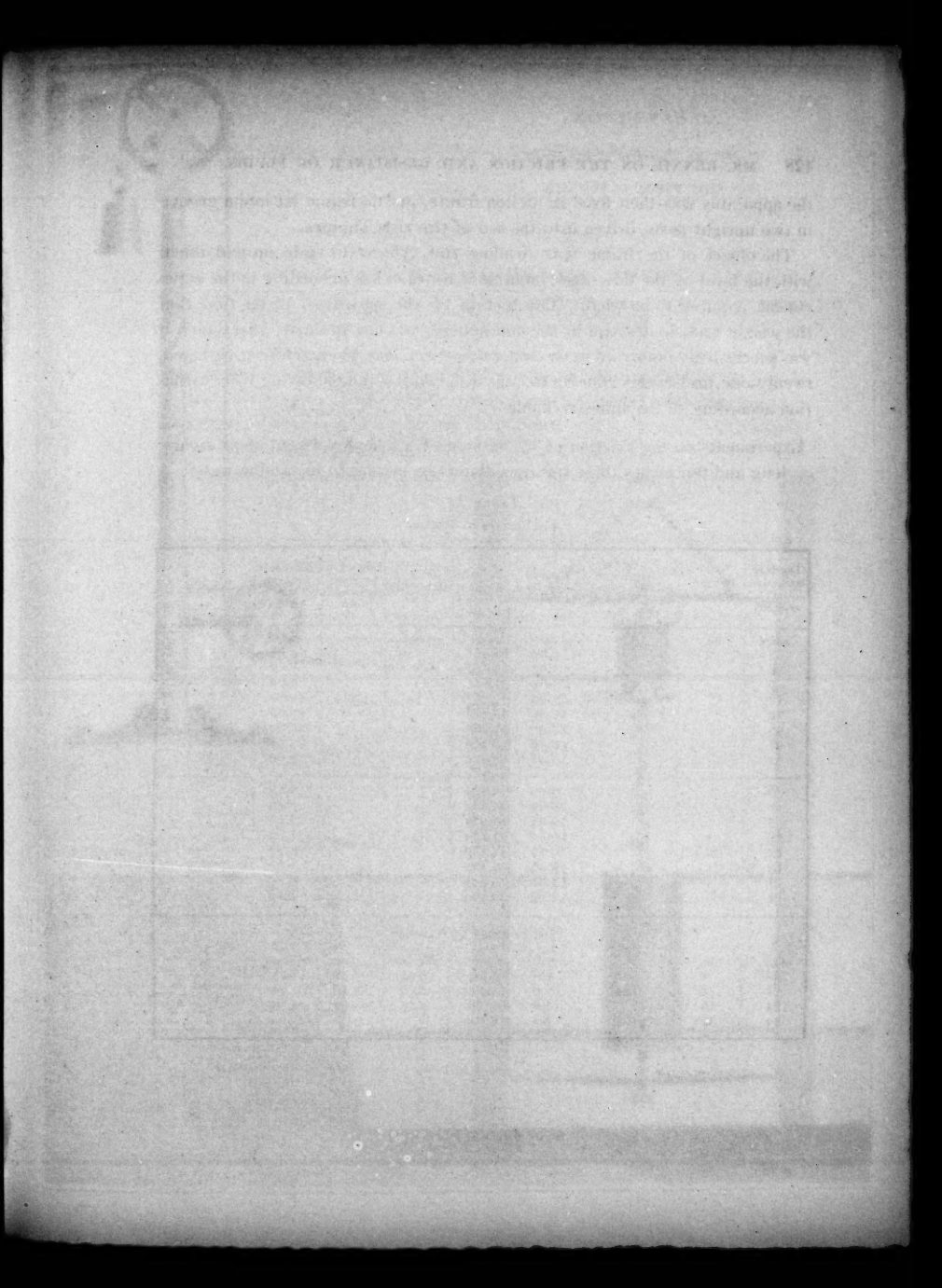
The object of the frame was to allow the cylinder to slide up and down with the level of the tide, and immerse it more or less according to the experiment required to be tried. The friction of the apparatus, or the time that the weight took to descend in the atmosphere, was first noticed; after which it was successively immersed in the water three, six, nine, twelve, fifteen, eighteen, twenty-one, and twenty-four inches, the difference of time showing the retardation according to the annexed Table.

Experiments on the Friction of the Surface of a Cylinder, twenty-four inches long and ten inches three quarters diameter, moving in air and in water.

TABLE I.

On Surfaces in Water.

| Depth of immersion of cylinder. | Weight suspended. | Number of revolutions of cylinder falling the whole height of 26 feet. | Time in descending in water. | Velocity of periphery per second in water. | Time in descending in air. | Velocity of periphery per second in air. | Difference between air and water. | Remarks. |
|---|---|---|---|---|---|---|---|---|
| inches. | lbs. | | seconds. | inches. | seconds. | inches. | seconds. | |
| 3 | | | 15.00 | 36.021 | | | 5.00 | |
| 6 | | | 18.00 | 30.017 | | | 8.00 | |
| 9 | | Sixteen turns in descending. | 25.00 | 21.612 | | | 15.00 | Resistance increased by surface with slow velocities, but not in the ratio of the surfaces. |
| 12 | 1 | Periphery moves through 540.32 inches. | 28.00 | 19.297 | 10 | 54.032 | 18.00 | |
| 15 | | | 32.00 | 16.885 | | | 22.00 | |
| 18 | | | 37.00 | 14.603 | | | 27.00 | |
| 21 | | | 40.00 | 13.508 | | | 30.00 | |
| 24 | | | 55.00 | 9.824 | | | 45.00 | |
| 3 | | | 9.00 | 60.035 | | | 4.00 | |
| 6 | | | 10.00 | 54.032 | | | 5.00 | |
| 9 | | | 10.50 | 51.459 | | | 5.50 | Resistance scarcely influenced by surface with increased velocities. |
| 12 | | | 10.50 | 51.459 | | | 5.50 | |
| 15 | 2 | Ditto. | 10.50 | 51.459 | 5 | 108.064 | 5.50 | |
| 18 | | | 10.50 | 51.459 | | | 5.50 | |
| 21 | | | 11.00 | 49.120 | | | 6.00 | |
| 24 | | | 11.00 | 49.120 | | | 6.00 | |
| On Velocities in Water. | | | | | | | | |
| inches. | lbs. | | seconds. | inches. | seconds. | inches. | seconds. | |
| 24 | 4 | | 8.0 | 67.54 | 2.45 | 196.48 | 5.15 | |
| 24 | 8 | Ditto. | 6.0 | 90.053 | 2.00 | 270.16 | 4.00 | Could not be tried. |
| 24 | 16 | | 4.0 | 135.08 | 1.50 | 360.21 | 2.50 | |
| 24 | 32 | | 2.5 | 216.128 | | | | |

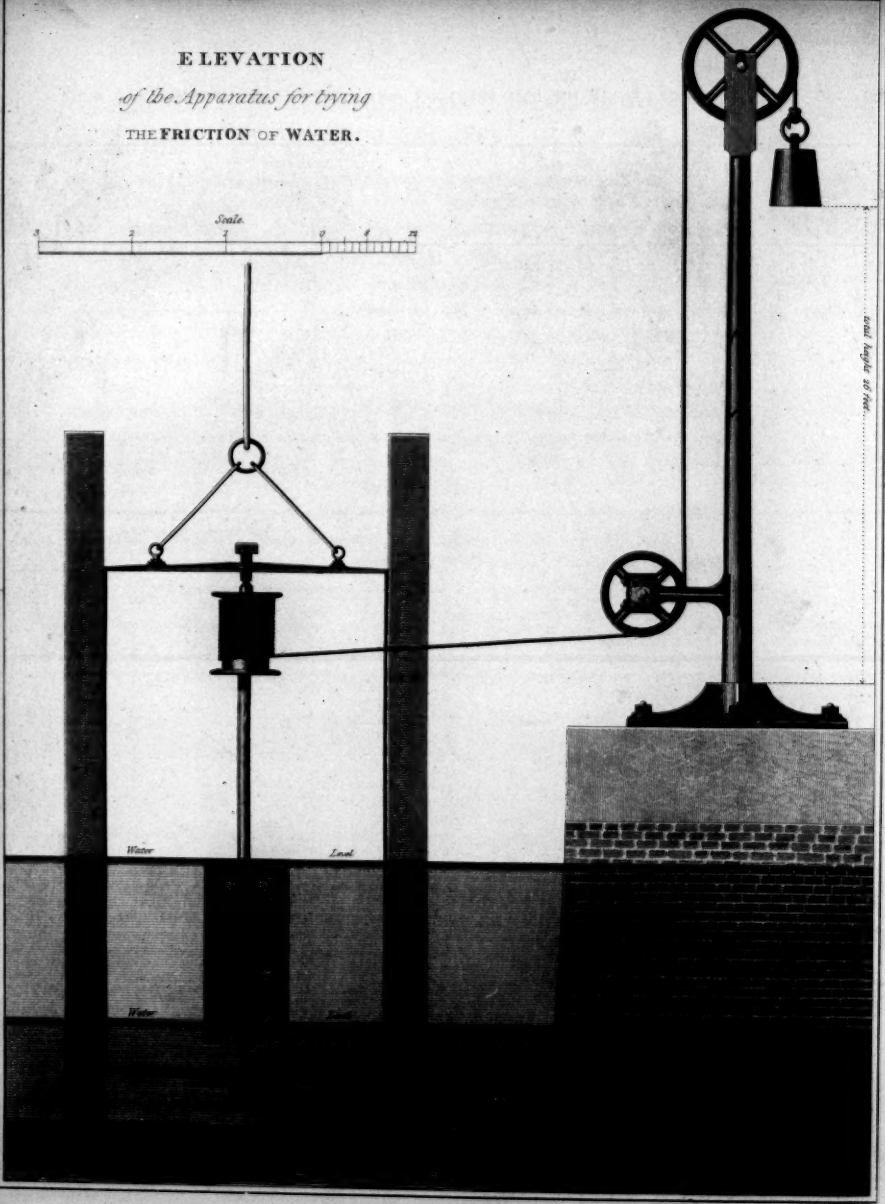


# ELEVATION

*of the Apparatus for trying*

THE **FRICTION** OF **WATER.**



*Conclusions.*

1. That the friction or adhesion of water against the surfaces of solids in motion, approximates the ratio of the surfaces with slow velocities; but that an increase of surface does not materially affect it with increased velocities.

2. That with equal surfaces the velocities do not seem to observe any fixed ratio, but approximate to the squares of the resistance.

With increased velocities the index of the power was found to be less than the duplicate ratio.

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To exemplify the result of the foregoing conclusion in a different way,—the cylinder was removed, and circular discs of iron, ten inches and three quarters diameter and one eighth of an inch thick, accurately adjusted to the spindle and polished, were substituted. The friction of the apparatus was again tried, and immersed in the river Thames, as before.

TABLE II. (See Plate XII. Fig. 2.)

Experiments on the Friction in Water of Circular Discs ten inches and three quarters in diameter and one eighth of an inch thick, revolving with the planes parallel to the horizon, and six inches apart.

| Number of discs. | Weight suspended. | Height fallen of weight. | Time of weight descending in water. | Velocity of periphery per second. | Time descending in air. | Velocity of periphery per second in air. | Difference. |
|---|---|---|---|---|---|---|---|
| lbs. | lbs. | | seconds. | inches. | seconds. | inches. | seconds. |
| | 1 | Twenty-five feet, mean circle 16.88 would move through 422 inches. | 10.00 | 42.200 | | | 8.00 |
| | 2 | | 5.00 | 84.400 | | | 3.00 |
| 1 | 3 | | 3.00 | 140.660 | 2 | 211 | 1.00 |
| | 4 | | 3.00 | 140.660 | | | 1.00 |
| | 6 | | 3.00 | 140.660 | | | 1.00 |
| | 1 | | 15.00 | 28.133 | | | 13.00 |
| | 2 | | 6.50 | 64.923 | | | 4.50 |
| 2 | 3 | Ditto. | 4.50 | 93.770 | 2 | 211 | 2.50 |
| | 4 | | 4.00 | 105.500 | | | 2.00 |
| | 6 | | 4.00 | 105.500 | | | 2.00 |
| | 1 | | 17.00 | 24.823 | | | 15.00 |
| | 2 | | 7.00 | 60.285 | | | 5.00 |
| 3 | 3 | Ditto. | 5.50 | 76.727 | 2 | 211 | 3.50 |
| | 4 | | 4.00 | 105.500 | | | 2.00 |
| | 6 | | 3.00 | 140.660 | | | 1.00 |
| | 1 | | 33.00 | 12.787 | | | 31.00 |
| | 2 | | 17.00 | 24.823 | | | 15.00 |
| 4 | 3 | Ditto. | 8.00 | 52.750 | 2 | 211 | 6.00 |
| | 4 | | 6.00 | 64.923 | | | 4.00 |
| | 6 | | 4.00 | 105.500 | | | 2.00 |

*Conclusions.*

That the friction or adhesion of water is not quite as the surfaces with slow velocities, being in the ratio of one to three instead of one to four, but diminishes rapidly, without observing any ratio in increased velocities*. Hence the resistance of a ship or vessel moving through the water, with an average or higher rate of velocity, forms an inconsiderable portion of the resistance resulting from the displacement of the fluid, and that the brightness observed on the copper of ships after a voyage, may be owing to other causes than the friction of the water simply.

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An experiment was made to ascertain the comparative resistance of a pipe revolving in water, and with water running through a pipe; when the resistance was found to be as the surfaces in slow velocities, but to diminish greatly, as before, in high velocities, without observing any fixed ratio.

The above conclusions are in contradiction to those of Coulomb, who did not find that pressure augmented the resistance, but states that the resistance is greater when the immersion is partial.

This apparatus being applicable to fluids generally, advantage was taken of it to ascertain the direct resistance of solids to fluids (see Plate XII.)†, by causing plates and globes to revolve in them, with their planes perpendicular to the plane of the horizon.

As the resistance of solids in fluids does not form the object of this paper, it will be unnecessary to introduce many detailed observations on the subject of these experiments at present, connected as they are with another branch of hydrodynamics. But as it is important to show the relation subsisting between the resistances of cohesion and impulse, I have ventured to detail the following experiments.

* The experiments of the Society for the Improvement of Naval Architecture show a decreased resistance with increased velocities.

† In this case, the number of particles struck will be diminished in the ratio of the radius to the sine of inclination; wherefore the resistance will be diminished in a duplicate ratio of the radius to the sine of inclination. But as the sines of inclination of the two plates are equal, the resistances will be equivalent to the area of one plate (moving perpendicularly to its planes) into the duplicate ratio of the velocity of its motion, and the density of the fluid.

## Table III.

### Experiments on the Rotations of Iron Discs and Wooden Balls moving in Air, with their planes perpendicular to the plane of the horizon.

| Weight suspended. | Height fallen. | Time in descending. | | | | | |
|---|---|---|---|---|---|---|---|
| | | Two circular discs 10½ inches diameter. Area 81 inches. | Velocity per second. | Two square fans. Area 81 inches. | Velocity per second. | Two wooden balls 10½ inches diameter. | Velocity per second. |
| lbs. | | seconds. | feet. | seconds. | feet. | seconds. | feet. |
| 2 | The spindle made 15.9 turns in falling 25 feet. Mean circle 51.88 would move through 68.67 feet. | 10.00 | 6.867 | 10.00 | 6.867 | 23 | 2.984 |
| 4 | | 6.00 | 11.445 | 7.00 | 9.810 | 13 | 5.282 |
| 9 | | 4.50 | 15.261 | 4.50 | 15.261 | 8 | 8.584 |
| 16 | | 3.00 | 22.891 | 3.25 | 21.130 | 7 | 9.810 |
| 20 | | 2.50 | 27.469 | 3.00 | 22.891 | 6 | 11.445 |

*Conclusions.*

1. That the resistances are as the squares of the velocity.
2. That the comparative resistances between discs and globes are as two to one nearly.

## Table IV.

### Experiments on the Resistance of Iron Discs and Wooden Globes revolving in Water.

| Weight. | Height fallen. | Time in descending. | | | | | |
|---|---|---|---|---|---|---|---|
| | | Two circular discs, 81 inches area. | Velocity per second. | Two square fans, 9 inches square, 81 inches area each. | Velocity per second. | Two wooden balls. Area 81 inches. | Velocity per second. |
| lbs. | | seconds. | feet. | seconds. | feet. | seconds. | feet. |
| 16 | The spindle made 15.9 turns in falling 25 feet. Mean circle 51.83 would move through 824.19 inches or 68.67 feet. | 63 | 1.09 | 53 | 1.29 | 15.00 | 4.57 |
| 20 | | 54 | 1.27 | 48 | 1.43 | 14.00 | 4.90 |
| 32 | | 43 | 1.59 | 40 | 1.71 | 10.50 | 6.59 |
| 40 | | 40 | 1.71 | 35 | 1.96 | 9.50 | 7.22 |
| 64 | | 30 | 2.28 | 28 | 2.45 | 8.00 | 8.58 |
| 256 | | 14 | 4.90 | 15 | 4.57 | 5.00 | 13.73 |

*Conclusions.*

1. That the resistances are as the square of the velocities.

2. That the mean resistances of circular discs, square plates, and globes in air, are as the numbers 25.180, 22.010, 10.627; and in water, 1.18, 1.36, 0.755; consequently the proportional resistance of air to water, with

Circular discs, is as 1 to 21.3
Plates and fans . . 1 to 16.2
Wooden balls . . . 1 to 2.2

*Note.*—A portion of the square fans, represented by the letters *a*, *b*, *c*, *d*, in Plate XII. fig. 1, and equal to one fourth of the area of each fan, was cut off, when the resistance was found to be the same as with the square fans.

## Experiments on the Quantities of Water discharged by Orifices and Tubes of different diameters and lengths, and at different altitudes.

The phenomena incident to spouting fluids are,

First, The inequality observed in the velocity of the particles comprised in every horizontal section parallel to the orifice.

Secondly, The contraction of the fluid vein beyond the orifice, and consequent diminution of discharge as compared with theory.

Thirdly, The inversion and changes in the sections of the fluid vein at different distances from the orifices.

All these phenomena have been noticed and recorded by various writers, and formulæ adapted to the different circumstances of the expenditure have been given. But neither Bossut nor Du Buat (the most accurate of writers) have recorded a continuous and systematic series of experiments upon the comparative expenditure of orifices and tubes under the circumstances of area, altitude, and length.

The apparatus with which these experiments were performed, consisted of a wooden cistern very accurately made, two feet square inside, and four feet four inches in height. The water was kept at a constant altitude by a regulating cock; and a float having an index attached to it enabled the observer to ascertain the exact height at which the water stood in the cistern above the centre of the orifice.

The orifices were accurately made by Dollond in brass plates one sixtieth of an inch in thickness. The plates were accurately adjusted to a hole in the

side of the cistern, and closed by a valve of brass ground to each of the plates. The valve was opened by a lever, and the time noted by chronometers.

The diameters of the tubes, from having been drawn on mandrils, were as accurate as possible; their diameters at the extremities were carefully enlarged, to prevent any wire edges from diminishing their sections; and one extremity of the tube being inserted into a block of hard wood fastened to the cistern, and the other stopped by a valve, the experiments were recorded as before.

### Table V.

Experiments on the Quantity of Water discharged by different-sized Orifices from a vessel kept constantly full and at different heights.

| Constant height of the surface of the water above the centre of the orifice. | Real time in discharging one cubic foot. | Theoretical time in discharging one cubic foot, $t = \frac{Q}{2A\sqrt{g}\,H^{\frac{1}{2}}}$. | Ratio of the theoretical to the real discharges. | Vena contracta. |
|---|---|---|---|---|
| **Circular Orifice made in a brass plate 1 inch diameter, $\frac{1}{10}$ inch thick.** | | | | |
| feet. | seconds. | seconds. | | |
| 4 | 19.50 | 11.4 | 1 : .584 | |
| 3 | 21 | 13.2 | 1 : .628 | Not accurately measured. |
| 2 | 26 | 16.1 | 1 : .619 | |
| 1 | 36 | 22.8 | 1 : .633 | |
| **Circular Orifice in a brass plate $\frac{3}{4}$ inch diameter, $\frac{1}{10}$th inch thick.** | | | | |
| 4 | 33 | 20.3 | 1 : .614 | At six tenths of an inch from the orifice, the diameter had contracted to 0.685 of an inch. |
| 3 | 37 | 23.4 | 1 : .632 | |
| 2 | 44 | 28.7 | 1 : .652 | |
| 1 | 63 | 40.6 | 1 : .644 | |
| **Circular Orifice in a brass plate $\frac{1}{2}$ inch diameter, $\frac{1}{10}$th inch thick.** | | | | |
| 4 | 73 | 45.7 | 1 : .626 | At half an inch beyond the orifice, the diameter contracted to 0.37 of an inch. |
| 3 | 83 | 52.8 | 1 : .636 | |
| 2 | 104 | 64.6 | 1 : .621 | |
| 1 | 144 | 91.4 | 1 : .634 | |
| **Circular Orifice in a brass plate $\frac{1}{4}$ inch diameter, $\frac{1}{10}$th inch thick.** | | | | |
| 4 | 276 | 182.9 | 1 : .662 | At a quarter of an inch beyond the orifice, the diameter contracted to one twentieth of an inch less than the orifice. |
| 3 | 320 | 211.3 | 1 : .660 | |
| 2 | 396 | 258.6 | 1 : .653 | |
| 1 | 545 | 365.7 | 1 : .671 | |

N.B. Each result shows the mean of four experiments.

### *Remarks.*

The phenomena relative to the form and direction of veins of spouting fluids, and the remarkable inversion of the fluid veins at certain distances from their orifices, have been so fully noticed in "Experiences sur la Forme et sur la Direction des Veines et des Courans d'Eau; par George Bidone: Turin, 1829," that it is unnecessary to state further than that they have been completely corroborated in the foregoing experiments.

## Table VI.

Experiments on the Quantities of Water discharged from Rectangular and Triangular Orifices in brass plates one sixtieth of an inch thick, and of equal areas, from a vessel kept constantly full and at different heights.

| Constant height of the surface of the water above the centre of the orifice. | Time in discharging one cubic foot. | Theoretical time in discharging one cubic foot, $\frac{Q}{2A\sqrt{gH}}$ | Ratio of real to theoretical discharge. | Form of orifice. |
|---|---|---|---|---|
| **Equilateral Triangle whose area is one inch, and angle uppermost.** | | | | |
| feet. | seconds. | seconds. | | Vena contracta about half an inch beyond the orifice; but the jet with the angles reversed, and taking the sides of the triangle, the jet afterwards expanded and lost its form.  |
| 4 | 15 | 8.9 | 1 : .593 | |
| 3 | 18 | 10.3 | 1 : .572 | |
| 2 | 22 | 12.7 | 1 : .577 | |
| 1 | 30 | 17.9 | 1 : .596 | |
| **Equilateral Triangle as before, with the angle downwards.** | | | | |
| 4 | 15 | 8.9 | 1 : .593 | Vena contracta the same as before, but the jet having its angle upwards, being the reverse of the former experiments.  |
| **Rectangular Orifice of one square inch.** | | | | |
| 4 | 15 | 8.9 | 1 : .593 | Vena contracta about three quarters of an inch beyond the orifice, when each angle of the jet took the place of a side thus, and dissipated in spray.  |
| 3 | 17 | 10.3 | 1 : .606 | |
| 2 | 20 | 12.7 | 1 : .635 | |
| 1 | 29 | 17.9 | 1 : .617 | |
| **Rectangular Orifice 2 inches long, $\frac{1}{2}$ an inch wide, having the long side parallel to the surface of the water.** | | | | |
| 4 | 15 | 8.9 | 1 : .593 | Vena contracta as before. Each angle of the jet took the place of a side.  |
| 3 | 17 | 10.3 | 1 : .606 | |
| 2 | 20 | 12.7 | 1 : .635 | |
| 1 | 29 | 17.9 | 1 : .617 | |
| **Rectangular Jet $1\frac{1}{2}$ inch long, $\frac{1}{4}$ wide, placed as before.** | | | | |
| 4 | 15 | 8.9 | 1 : .593 | Vena contracta as before.  |
| 3 | 17 | 10.3 | 1 : .606 | |
| 2 | 19 | 12.7 | 1 : .668 | |
| 1 | 27 | 17.9 | 1 : .663 | |

### *Remarks.*

That with equal areas, the expenditure by different orifices, whether circular, rectangular, or triangular, is nearly the same, the increase being in favour of rectangular orifices.

## Table VII.

Experiments on the Quantity of Water discharged by Cylindrical Glass Orifices and Tubes, from one inch in length to one foot, and of different diameters, from a vessel kept constantly full, and at different heights.

| Constant height of the surface of water above the centre of orifice. | Time in seconds in discharging one cubic foot. | | | | Remarks. |
|---|---|---|---|---|---|
| | 1 inch. | ½ in. | ¼ in. | ⅛ in. | |
| feet. | | | | | |
| 4 | 11.5 | 24.5 | 55 | 145 | In comparing these experiments with the time and quantity discharged by plate orifices, there is a diminution of time, and an increased discharge of from ½ to ¼. |
| 3 | 15.0 | 28.5 | 63 | 157 | |
| 2 | 17.5 | 35.0 | 77 | 205 | |
| 1 | 25.0 | 53.0 | 110 | 297 | |
| From Glass Tubes one foot long. | | | | | |
| | 1 inch. | ½ in. | ¼ in. | ⅛ in. | |
| 4 | 14.0 | 30 | 63 | 200 | Shows an increase of time and a diminution of discharge in the ratio of from ½ to ¼. |
| 3 | 17.0 | 33 | 73 | 227 | |
| 2 | 21.5 | 40 | 88 | 283 | |
| 1 | 30.0 | 58 | 130 | 410 | |

### *Conclusions.*

1. That the quantities discharged in equal times by orifices and additional tubes, are as the areas of the orifices.

2. That the quantities discharged in equal times by the same additional tubes and orifices under different heads, are nearly as the square roots of the corresponding heights.

3. That the quantities discharged in equal times by the different additional tubes and orifices under different heights, are to one another in the compound ratio of the areas of the apertures, and of the square roots of the heights.

From the foregoing experiments the mean coefficient for altitudes of 4 feet with the circular orifices, is ..... 0.621

but with altitudes of 1 foot the coefficient is ..... 0.645

with triangular orifices at 4 feet altitude ..... 0.593

with triangular orifices at 1 foot altitude ..... 0.596

with rectangular orifices at 4 feet altitude ..... 0.593

with rectangular orifices at 1 foot altitude ..... 0.616

Hence, allowing for the inaccuracies incident to experiments of this nature, we may safely adopt Messrs. Prony and Bossut's coefficients for altitudes of 4 feet ..... 0.621

—— —— —— 1 foot ..... 0.619

In the case of additional tubes of glass the coefficient is much higher than Bossut's, which was for 4 feet 0.806, and 1 foot 0.817.

*Note.*—Vide Venturi and Eytelwein's experiments.

Let A = area of orifice in square feet.
$d$ = diameter of orifice if circular.
H = altitude of the fluid in feet.
T = time.
$g$ = gravity in one second.
Q = quantity of water in cubic feet.

According to Bossut's experiments $Q = 0.61938\,A\,T\,\sqrt{2\,g\,H}$.

And as $2\,g$ is a constant quantity, and is equal to 7.77125, we have $Q = 4.818\,A\,T\,\sqrt{H}$ for orifices of any form, substituting $d$ if circular, or $Q = 3.7842\,d^2\,T\,\sqrt{H}$.

From the second of these equations we obtain

$$A = \frac{Q}{4.818\,T\,\sqrt{H}} \quad T = \frac{Q}{4.818\,A\,\sqrt{H}} \quad \text{and} \quad H = \frac{Q}{(4.818\,A\,T)^2}.$$

For additional tubes the equation will stand thus: $Q = 0.81\,A\,T\,\sqrt{2\,g\,H}$; but since $2\,g$ is constant, and is 7.77125, we have $Q = 4.9438\,d^2\,T\,\sqrt{H}$, from which we deduce

$$d = \sqrt{\frac{Q}{4.9438\,T\,\sqrt{H}}} \quad T = \frac{Q}{4.9438\,d^2\,\sqrt{H}} \quad H = \frac{Q}{(4.9438\,d^2\,T)^2}.$$

## Table VIII.

**Experiments on the Friction or Quantity of Water discharged by Leaden Pipes of different diameters and lengths, from a vessel kept constantly full, and at different heights.**

| Pipes 15 feet long each, straight. | | | | | |
|---|---|---|---|---|---|
| Constant height of the surface of the water above the centre of the pipe. | Time in discharging one cubic foot. | | | | Remarks. |
| | 1 inch. | ½ in. | ¼ in. | No leaden pipes to be had ½ bore. | |
| feet. | seconds. | seconds. | seconds. | | |
| 4 | 28 | 54 | 143 | | The time in discharging one cubic foot is nearly double the time occupied by glass tubes of equal lengths and areas. |
| 3 | 33 | 63 | 164 | | |
| 2 | 41¼ | 79 | 208 | | |
| 1 | 61¼ | 117 | 312 | | |

## Table IX.

Experiments on the Quantities of Water discharged by Leaden Pipes $\frac{1}{2}$ inch bore, but of different lengths from one foot to thirty feet in length.

| Glass tubes 1 inch long, $\frac{1}{2}$ inch diam. | | Brass orifice $\frac{1}{2}$ diam. | 1 foot long. | 3 ft. 9 in. | 7 ft. 6 in. | 11 ft. 3 in. | 15 ft. | 30 ft. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|
| feet. | seconds. | seconds. | seconds. | seconds. | seconds. | seconds. | seconds. | seconds. | |
| 4 | 55 | 73 | 55 | 78 | 102 | 122 | 143 | 203 | The ratio of discharge by glass tubes with pipes of 30 feet long, is as 1 : 4 } nearly. Ditto with brass orifices, is as 1 : 3 } |
| 3 | 63 | 83 | 63 | 92 | 120 | 145 | 164 | 240 | |
| 2 | 77 | 104 | 93 | 113 | 151 | 184 | 208 | 303 | |
| 1 | 110 | 144 | 133 | 170 | 226 | 278 | 312 | 450 | |

### *Conclusions on Pipes of different lengths.*

That the expenditures of water by pipes of equal diameters but of unequal lengths and under different altitudes, are nearly as follow.

The length being as 30 to 1, the expenditures are as 3.7 to 1
Do. . . . 8 to 1 . . . do. . . . 2.6 to 1
Do. . . . 4 to 1 . . . do. . . . 2.0 to 1
Do. . . . 2 to 1 . . . do. . . . 1.4 to 1

The discharges by glass and leaden tubes are nearly alike. The length of a pipe may be increased from 3 to 4 feet without diminishing the discharge as compared with the plate orifices.

## Table X.

### Experiments on Leaden Pipes with Flexures.

The straight pipe of $\frac{1}{2}$ an inch bore, on which the preceding experiments were made, was carefully bent into one, two, and fourteen semicircular bends respectively, each of $7\frac{1}{2}$ inches in the semidiameter, and two of $\frac{1}{4}$th part of a circle of $3\frac{1}{4}$ inches radius. One end of the pipe was fixed in the wooden orifice as before, and the following are the results.

| Pipe 15 feet long, $\frac{1}{2}$ inch bore, with one semicircular and two $\frac{1}{4}$-circle bends. | | | |
|---|---|---|---|
| Constant height of the surface of the water above the centre of the orifice. | Time in discharging one cubic foot by a pipe with three bends. | Time discharging one cubic foot by a straight pipe. | Remarks. |
| feet. | seconds. | seconds. | |
| 4 | 147 | 143 | The position of the bends, whether vertical or horizontal, at either extremity of the pipe, does not affect the result. |
| 3 | 175 | 164 | |
| 2 | 213 | 208 | |
| 1 | 316 | 312 | |

| Pipe 15 feet long, $\frac{1}{2}$ inch bore, with 14 semicircular and two $\frac{1}{4}$-circle bends. | | | |
|---|---|---|---|
| feet. | seconds. | seconds. | |
| 4 | 162 | 143 | The expenditure is diminished by the bends from $\frac{1}{4}$ to $\frac{1}{5}$, which represents the friction of the pipe. |
| 3 | 200 | 164 | |
| 2 | 247 | 208 | |
| 1 | 351 | 312 | |

### *Results.*

1. That with one semicircular and two $\frac{1}{4}$ of a circle bends, as compared with a straight pipe of equal length and bore, the resistance varies from $\frac{1}{7}\frac{1}{7}$th to $\frac{1}{7}\frac{1}{0}$th part of the resistance of the straight pipe.

2. That with fourteen semicircular and two quarter of a circle bends, the resistance varies from $\frac{1}{7}\frac{1}{7}$th to $\frac{1}{7}\frac{1}{5}$th of the resistance of a straight pipe.

3. That the increased number of bends does not increase the resistance in the ratio of the number of bends, but merely shows an increased resistance, as compared with the four bends, of $\frac{1}{7}\frac{1}{5}$th to $\frac{1}{7}\frac{1}{5}$th.

## Table XI.

Experiments on the Discharge of Water by Leaden Pipes of $\frac{1}{2}$ an inch bore, 15 feet long, but bent in the forms of from one to twenty-four right-angled elbows, each side being $6\frac{3}{4}$ inches long.

| Height of the surface of the water above the centre of the orifice. | One right angle $8\frac{1}{4}$ inches from the end of the pipe. | Straight pipe 15 feet long. | Twenty-four right angles. | Remarks. |
|---|---|---|---|---|
| feet. | seconds. | seconds. | seconds. | |
| 4 | 180 | 143 | 395 | In the first three experiments we have a diminution of expenditure in the ratio of $2\frac{1}{4}$ to 1, and in the last experiment as 3 to 1 nearly. |
| 3 | 214 | 164 | 465 | |
| 2 | 246 | 208 | 584 | |
| 1 | 371 | 312 | 872 | |

### *Conclusions.*

From the foregoing experiments with one rectangular pipe, it would be reasonable to conclude that the diminution of discharge would be as the number of right angles; but comparing the expenditure by one right-angled pipe with the expenditure of a pipe with twenty-four right angles, the difference is only in the ratio of about two to one.

### *General Remarks on the Expenditure of Horizontal and Bent Pipes.*

Formulæ adapted to the different circumstances of the motion of water in pipes and conduits have been given by various authors.

By some, the retardations were supposed to be in the inverse ratios of the squares of the lengths of the pipes; and by others, to be represented by a certain portion of the altitude of the reservoir above the centre of the pipe, the resistance being directly as the length and circumference of the pipe, and inversely as the area of the section.

M. Girard, in his beautiful experiments*, conceived the resistance to be compounded of the first and second powers of the velocity. So that, deducing the values from Dubuat's experiments, and expressing the resistance due to cohesion by $R x U$, R being the quantity to be obtained by experiment, and making

* Memoires des Scavans Etrangers.

the resistance due to the asperities equal to $R x U^2$, the sum of the resistance is $R (U + U)^2$.

M. Prony, applying his profound acquirements to the solution of all the cases of preceding authors, deduced from a selection of upwards of fifty experiments the following simple formula: $U = 26.79 \sqrt{\frac{D Z}{\lambda}}$;

U being the mean velocity of the section of the pipe;
D the diameter of the pipe;
Z the altitude of the water;
$\lambda$ the length of the pipe:

from which it appears that the velocity is directly in the compound ratio of the square roots of the diameter of the pipe and head of water, and inversely as the square roots of the length of the pipe; that is, for any given head of water and diameter of pipe, the velocity is inversely as the square root of the length of the pipe.

If we compare these results with those of Dubuat, Girard, and others, they approximate very nearly to each other.

In general, if we incline a pipe to an angle of about $6\frac{1}{2}$ degrees, or one ninth of its length, the discharge will be nearly equal to the discharge by additional tubes. The charge necessary to express the mean velocity of water issuing from straight pipes is by some authors equal to $\frac{V^2}{478}$ *; Dr. Young makes it $\frac{V^2}{550}$: the diminution of expenditure depending upon the contraction of the fluid vein and the friction of the pipe.

The change occasioned by bends and angles in the direction of the fluid vein tends to diminish the velocity in a very remarkable manner.

Dubuat undertook several experiments upon this subject, but the formula proposed by him does not solve the difficulty, where $\frac{V^2 S^2}{m}$ gives the resistance due to one bend, V being the velocity, S the sine of incidence or reflection, and $m$ a constant quantity determined by Dubuat to be 2998.50.

Now although it is reasonable to suppose that the resistance should be proportionable to the squares of the sines of the angles of incidence, yet as all

* Dubuat and Langsdorf.

the particles of the fluid vein are not reflected in the same angle, and as a considerable portion of the velocity is destroyed by the first angle or bend the fluid meets with in the pipe, M. DUBUAT's theory is fundamentally erroneous, the more especially as he has rejected more than one half of the twenty-five experiments mentioned by him. Dr. YOUNG's suppositions, of the resistance being as the angular flexure and the power of the radius, of which the index is $\frac{7}{5}$, are equally erroneous, as is evinced by the foregoing experiments.

In conclusion, it is evident that the subject of friction admits of an immense variety of applications. To determine the measure of the resistances experienced by vessels and floating bodies in their motion through fluids; the law of the retardations of rivers, and the cause of the obstructions presented to the waves of the ocean in the slopes assumed by its shores; the equilibrium of earths, and their connection with solids and fluids,—all of them are questions of the utmost importance in the economy of nature, and their solution can only be attained by an accumulation of facts.

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N.B. Since the foregoing paper was presented to the Royal Society, an abstract of an extensive series of experiments on the expenditure of water through rectangular orifices of large dimensions, has been submitted to the French Academy by Messrs. PONCELET and LESBROS, of the Corps de Genie at Metz; and as these experiments were undertaken by order of the French government, no expense was spared to have them made as extensive as possible. Their objects were principally to ascertain the exact measure of the coefficient of contraction and the forms of the fluid veins under different altitudes and areas.

The results of which are:—

That with an orifice of 20 centimetres square, the coefficient is 0.600 under altitudes of 1 metre 68 centimetres. But when the altitude was reduced to four or five times the opening of the orifice, the coefficient increased to 0.605, but again diminished rapidly as the altitude diminished, to 0.593.

That with orifices of smaller dimensions, i. e. from 10 to 5 centimetres square, the same law was observed, the coefficients being respectively 0.611,

0.618, and 0.611, for opening of 10 and for 5 centimetres, 0.618, 0.631, 0.623; and for orifices of less dimension, the coefficient continually increased up to 0.698.

That for water running over weirs, the mean coefficient was 0.400, which differs very little from that of BIDONE.

Hence we see little reason to deviate from the coefficients already given.

## XXIV. *Further Experiments with a new Register-Pyrometer for Measuring the Expansion of Solids. By* J. Frederick Daniell, *Esq. F.R.S. Professor of Chemistry in King's College, London.*

Read June 16, 1831.

In my former communication on a new Register-Pyrometer, which has been honoured with a place in the Philosophical Transactions for 1830, I stated that I hoped, at some future period, to be able to lay before the Society the results of some experiments upon the dilatation of metals to their melting points; and I now purpose to redeem this pledge.

My previous observations upon the subject of expansion, were directed chiefly to the object of establishing what degree of confidence might be reposed in the instrument as a measure of temperature; and I was able, I trust, to exhibit such an accordance between the measures which it had afforded and those of the best experimenters, long previously obtained with various metals to the boiling point of water, as fully to establish its sufficient accuracy. The comparison however which I most relied upon, was with the experiments of MM. Dulong and Petit, upon the expansion of platinum and iron to the high temperature of 572° Fahr.; and as this is a point of fundamental importance, I shall still further strengthen it by a comparison with the results obtained by the same distinguished philosophers with copper, the only other solid metal to which they extended their inquiries.

Previously to this, I trust it may not be thought tedious, if I briefly relate the results of some trials for obtaining registers of uniform composition, which might preclude the necessity of determining the rate of expansion in each individual instance.

Exp. 23. For this I had recourse to Wedgwood's ware, of which I obtained some bars carefully constructed and highly baked for the purpose. The expansion of these I found precisely equal to that of platinum; so that when

the register was immersed in boiling mercury, the index was found not to have moved. When a bar of iron was substituted for that of platinum, the arc measured was 1° 7′.

With black-lead the same expansion gave a measure of 2° 49′, from which if we deduct the expansion of platinum in black-lead . 1 45

the remainder . . . . . 1 4 is sufficiently near to confirm the result.

Exp. 24. My next trial was with registers of black-lead of various and known mixtures of plumbago and Stourbridge clay. Four fifths proportion of the former to one fifth of the latter produced a composition which was too tender for the purpose; but a mixture in the proportion of three fourths to one fourth formed a ware of a fine, even texture; whose expansion was very equal, and not exceeding the least of those which I had formerly tried.

Three different registers of this composition afforded me the following measures of the expansion of a platinum bar to the boiling point of mercury.

1° 45′
1 42
1 38

To which I may add a fourth, which gave for the expansion of an iron bar to the same point an arc of 2° 42′, which is equivalent to 1° 40′ for a platinum bar. For all common purposes, therefore, the mean expansion of 1° 42′ might have been adopted without any serious error in the final results. In investigations, however, which require the utmost precision, I still think it advisable to fix the expansion of each register by experiment.

Exp. 25. A bar of copper was adjusted in one of the registers and exposed, in the manner formerly described, to boiling mercury; the arc measured on the scale was 4° 10′, equivalent to an expansion of .03633.

Let us now compare this result with the determination of MM. Dulong and Petit, as we formerly did the expansions of platinum and iron.

The expansion of Copper.

Length of Bar.

From 32° to 212° = .0017182 × 6.5 . . . . . = .01116830
From 392° to 572° = .0018832 × 6.5 . . . . . = .01224080

.02340910

From 212° to 392° = Mean of the above . . . . = .01170455

Total expansion from 32° to 572° . . . . . = .03511365
Add for the expansion from 572° to 660°,
the temperature of boiling mercury, calculated at
the highest rate:—
180° : .0018832 :: 88° : .00920675 . . . . . = .00920675

.04432040

Deduct expansion for 32°, the experiment with the
pyrometer having commenced at 64° . . . . . = .00305457

Calculated at the lowest rate:—
180 : .0017182 :: 32° : .00305457
Real expansion of the bar by DULONG and PETIT . = .04126583

If from the real expansion thus obtained . . . . . .04126
We deduct the apparent expansion obtained by the pyrometer .03633

The remainder .00493

will be the expansion of the black-lead.

We thus obtain the expansion of 6.5 inches of black-
lead ware,
from 64° to 660° by Platinum bar . . . . . 00421
by Iron bar . . . . . 00457
by Copper bar . . . . . 00493

Mean .00457

in which the extreme results differ from the mean not .0004 inch, or one fourteenth of the whole.

When we take into consideration the great difference in the total expansion of these three metals, as well as the differences in their several rates of increase

with the increasing temperature, such an accordance appears to me to be perfectly decisive of the accuracy of the pyrometer.

It will be unnecessary for me to trouble the Society with the details of the experiments by which I determined the expansion of several other metals to the boiling point of mercury; it will be sufficient to state the results in a tabular form. I thought that it would add much to the interest of the determination of the total expansion to the fusing points, to determine previously the expansion of each to the points of boiling water and boiling mercury; that any alteration in the rates of expansion between these points might be detected.

I must, however, make a few observations upon the general method which I adopted to insure an accurate determination of the former.

Exp. 26. Judging from the action of the pyrometer at lower heats, I expected that the index would continue to be thrust forward by the progressive expansion of any bar of metal, till its cohesion gave way and it assumed the fluid form; and consequently that a register would be obtained of its maximum dilatation: but the difficulty consisted in applying the heat so equally that one part should not melt before another. The arrangement which I finally adopted to secure this purpose, and which was found to answer perfectly, was as follows. In the laboratory of the Royal Institution there is an excellent wind-furnace, from which proceeds a lateral horizontal flue, along which a flame may be drawn with any required degree of force. Into this flue open two muffle-holes, which give a complete view and command of the interior. From the equality of the draught, regulated by a register, the whole of this chamber may be kept at a low red, or an intense white, heat, by a proper management of the fuel in the body of the furnace.

The registers of the pyrometer were prepared for the experiment by drilling three holes on their under sides, communicating with the cavities in which the bars were placed; one at each extremity, and one in the centre. This was done for the purpose of allowing a vent for the melted metal, and to afford some criterion of the equality of the heat, by the time at which the metal ran from the different apertures. When the bar was properly adjusted in the register, it was carefully placed in the hot air-chamber, in a horizontal position, supported at each end by a small piece of brick, at a proper distance from the

body of the fuel, accordingly as a greater or less degree of heat was required. The muffle-holes were then closed with their stoppers; all but a narrow slit, through which the progress of the heating and the flow of the metal could be observed. The equality of the heat could be very accurately ascertained by the uniform colour of the register as it became red; and any irregularity could easily be corrected by advancing one or other end more towards the fuel. In this manner I succeeded in obtaining very satisfactory results; except in the case of gold; and this metal requiring for its fusion rather more heat than I could at the time command in the air-chamber, I laid the register upon the fuel in the body of the furnace, and it thus became only partially melted, and half the bar remained in the solid state. The amount of the expansion indicated is therefore evidently deficient, and must be discarded from the table. A similar accident happened once with brass; but this I have been able to rectify by subsequent trials.

I shall now arrange the results of my experiments in two tables:—the first showing, in arcs of the scale, the expansion of pure metals from 62° F. to 212°, 662° F., and their respective melting points; and the second exhibiting the expansion of certain alloys to the same points.

The bars were in all cases of the same length of 6.5 inches.

## Table XIII.

Showing the progressive expansion of the following pure metals to their melting points.

| From 62° | to 212° | to 662° | to Melting Point. |
|---|---|---|---|
| | ° ′ | ° ′ | ° ′ |
| Tin | 0 55 | ..... | 2 30 |
| Lead | 1 33 | ..... | 6 17 |
| Zinc | 1 40 | 5 50? | 8 44 |
| Silver | 0 59 | 4 9 | 13 45 |
| Copper | 0 45 | 4 10 | 16 0 |
| Gold | 0 35 | 3 11 | (7 51 not correct) |
| Cast Iron | 0 29 | 2 25 | 9 47 |

### Table XIV.

Showing the progressive expansion of the following alloys to their melting points.

| From 62° | to 212° | to 662° | to Melting Point. |
|---|---|---|---|
| | ° ′ | ° ′ | ° ′ |
| Brass. Common | 0 54 | 4 42 | (8 41 not correct) |
| Brass. Copper ½, Zinc ½ | 1 9 | 4 51 | 13 39 |
| Brass. Copper ½, Zinc ½ | 1 27 | 5 3 | 15 34 |
| Bronze. Copper ¼, Tin ¼ | 0 52 | 3 37 | 9 49 |
| Bronze. Copper ½, Tin ½ | 0 54 | 4 11 | 10 16 |
| Bronze. Copper ½, Tin ½ | 0 58 | 4 44 | 10 53 |
| Bronze. Copper ½, Tin ½ | 1 0 | 4 7 | 4 7 ? |
| Pewter. Lead ½, Tin ½ | 1 5 | ..... | 2 28 |
| Type Metal. Lead and Antimony | 1 5 | ..... | 3 13 |

The first remark which I shall make upon these tables regards the fusing points of the pure metals. Having ascertained for each the expansion due to certain definite increments of temperature, and the utmost expansion which they undergo to their fusing points, it is clear that, had their expansion been equal for equal increments, we might have determined the true temperature of their melting points from these data. As it is, even, knowing something of the limits of error introduced into such a calculation by the increased rate of expansion at the upper part of the scale, and the direction in which it ought to affect the result, we may draw some important inferences with regard to the correctness of the determinations derived from other means. The following Table exhibits the results of such a calculation, compared with the melting points previously determined.

### Table XV.

Fusing points of metals derived from their expansions to 212° and 662° supposed equable.

| | From 212° rate. | From 662° rate. | Real Temperature. |
|---|---|---|---|
| | ° | ° | ° |
| Tin | 471 | ..... | 442 by Thermometer. |
| Lead | 670 | ..... | 612 by Thermometer. |
| Zinc | 848 | 960 ? | 773 by Pyrometer. |
| Silver | 2159 | 2049 | 1873 by Pyrometer. |
| Copper | 3262 | 2366 | 1996 by Pyrometer. |
| Cast Iron | 3096 | 2489 | 2786 by Pyrometer. |

Now by these results, the accuracy of the pyrometer may, again, be placed beyond doubt, in a manner which was perfectly unforeseen at the time of instituting the experiments.

In the first place we have two metals, tin and lead, whose melting points being within the temperature of boiling mercury, have been accurately determined by the common thermometer. Upon calculating the same points from their several expansions to boiling water, measured by the pyrometer, upon the supposition that they maintain the same rate to their points of fusion, the temperature of the first comes out 29°, and of the second 58°, higher: that is to say, the rate of expansion of these two metals increases with the increase of temperature, as has been found to be the case with platinum, iron and copper, by the experiments of MM. Dulong and Petit. It is worthy of remark, that this increased rate in tin is equivalent to 29° in about 200°, and in lead to 58° in about 400°, above the boiling point of water. These results therefore indicate a very close agreement between the thermometer and pyrometer.

2ndly. The melting point of the next metal, zinc, is one of those which has been determined by immersion of the pyrometer into it, when it was in the act of fusion. Its temperature, so determined, falls short of the same point, calculated from the expansion supposed equal, by 75°. This again indicates an expansion increasing at nearly the same rate (75° in 560°), as in the preceding instances of tin and lead. I pass over at present the result obtained by calculating from the expansion to the boiling point of mercury, as it presents an anomaly upon which I shall presently make some observations.

3rdly. The melting point of silver, determined in the same way by immersion, differs from that calculated from expansion in the same direction; and the difference (286° in 1660°) is nearly in the same proportion. The calculation from the rate of expansion to the boiling point of mercury comes much nearer to the melting point directly determined, and only differs from it 176°: proving that the rate of expansion increases with the increasing temperature.

4thly. A similar comparison instituted with copper presents us with a rate of expansion increasing much more rapidly than in the preceding instances; so that the melting point, calculated from the expansion to boiling water, differs from the true melting point no less than 1266°. Taking the rate of expansion to boiling mercury, the difference is reduced to 370°. And here again I may

refer to the experiments of MM. Dulong and Petit in confirmation of the result; for they found that the temperature indicated by the expansion of a rod of copper was 50° Fahr. higher than the true temperature at 572° Fahr.

5thly. The interesting nature of the results which I obtained with iron, and the peculiar difficulties in arranging the experiments from which they were derived, will, I trust, excuse my entering more into their details than I have thought necessary in the preceding instances. I have already given the expansion of wrought iron to the temperatures of boiling water and boiling mercury, and shown that the measures obtained with the pyrometer agree essentially with those determined by very different means by MM. Dulong and Petit. I have also proved that the melting points of gold and silver, determined by the expansion of the same bar of iron, agreed very closely with the same points determined by the expansion of platinum. I was extremely anxious to complete this series of experiments by measuring the expansion of iron to its melting point. For this purpose I had a small bar of iron cast from the best gray iron, and afterwards cleaned of all oxide and reduced to the size of the other bars employed by filing. Upon measuring its expansion to the temperatures of boiling water and boiling mercury, I found the arcs upon the scale respectively 0° 29′ and 2° 25′; and this being considerably less than what I had obtained with the bar of wrought iron, I repeated the experiment with the latter in the same register that I had employed for the former, and obtained the measures of 0° 35′ and 2° 44′—nearly agreeing with the previous determination: so that there can be no doubt that cast iron expands less than wrought iron, though the rate of increase for the higher temperature appears to be the same in both.

I now arranged the two bars in two registers; and having strongly heated the furnace and filled the air-chamber itself with coke, I cleared out a space in which they could be placed, without coming in contact with the fuel on each side of them. Their two ends rested on pieces of fire-brick; the wrought iron was placed lowest, and, the thickness of the register, in advance of the cast iron; which was placed about two inches higher. The apertures were now all closed, and the draught increased to the utmost. At the expiration of a quarter of an hour the register with the cast iron was removed with a pair of tongs; and the metal upon lifting it, immediately flowed out at the two end

holes. The register, with the wrought iron was then taken out. The bar of the latter was found perfect, without any signs of oxidation or fusion.

The arc measured of the cast iron was . . . . 9° 47′

The arc of the wrought iron . . . . . 7° 56′

I had some reason to think that the register, with the wrought iron bar, had not been exposed so fully to the heat as that with the cast iron: for, although placed slightly in advance of the latter towards the body of the furnace, it was not raised so high from the floor of the flue, which probably had a cooling influence; and as the flame was drawn upwards, it must have struck with greater force upon the higher register. I therefore replaced the wrought-iron bar in the register, and put it exactly in the position previously occupied by the cast iron; it was then covered with charcoal, and the fire urged to the utmost. At the expiration of twenty minutes it was removed: the bar was found uninjured, with a white metallic lustre, except over the apertures, where it was blue, and perfectly free from oxide. The arc now, however, measured 11° 16′.

Now from these experiments there are four ways of approximately determining the temperature of melting cast iron.

1st. By taking the expansion of cast iron to its melting point, and calculating from the expansion for 150° to the boiling point of water, upon the supposition that the same rate is maintained, and adding the initial temperature of 60°, we obtain 3096°.

2ndly. By calculating from the expansion of the same bar for 600° to the boiling point of mercury, supposed equal, we obtain 2489°.

3rdly. By assuming the expansion of a bar of wrought iron, at the point of melting cast iron, and calculating from the expansion of the same bar for 150° to the boiling point of water, we obtain 2957°.

4thly. By calculating from the expansion of the same bar for 600° to the boiling point of mercury, supposed equal, we obtain 2533°.

It is remarkable that the mean of these four determinations is 2768°; for it will be remembered that the corrected temperature, which I deduced from the expansion of a platinum bar plunged into melting cast iron, was 2786°.

It may be observed, that in both cast iron and wrought iron, the calculation from the rate of expansion to the boiling point of water gives a temperature higher than the true; and that, in both, the calculation from the point of boil-

ing mercury affords a result lower than the true. This might afford some grounds for conjecturing that, although the rate of expansion evidently increases beyond the temperature of boiling water, it does not continue to increase to the end; but there is another inference from the fact, which I am rather inclined to adopt.

In calculating the temperature of melting cast iron, from the expansion of the platinum bar, I applied a correction, upon the supposition that the same rate of increase of expansion which was exhibited by platinum between the boiling points of water and mercury continued to the higher degrees; whereas there is great reason to suppose that the rate must be an increasing one; and, although this might not sensibly affect the final result of the comparatively low temperature of melting silver, the calculation of the temperature of melting iron, which is more than one third higher, would be sensibly affected by it. I think it therefore extremely probable that the true temperature of melting cast iron is below 2786°.

The consistency of these results will, I trust, remove any doubts as to the competency of the pyrometer to determine fixed and comparable points of very high temperatures, and induce those connected with arts and manufactures to introduce its use, for the purpose of ascertaining many questions of the highest interest, both to practical and theoretical science. The experiments just detailed upon bars of wrought iron remove even the only trifling objection which could be brought against its general use; namely, the expense of a platinum bar: for it is quite proved that a bar of wrought iron is sufficient for every practical purpose, and it affords the important additional advantage of a much more open scale.

I proceed now to remark that zinc, as well as iron, appears by the Tables to present an exception to the law of an increasing rate of expansion with increasing temperature; the expansion for the 600° to boiling mercury not being so much as four times that for the 150° to boiling water. I cannot, however, from some peculiar circumstances attending the experiment, place entire confidence in the result. When, after boiling in mercury, the register was opened, the vapour was found to have gained admittance, and to have acted upon the zinc. It was firmly fixed in the cavity, and was not removed without considerable difficulty and piecemeal. At its upper end, the bar was

reduced almost to a point, and was very considerably thickened at its lower end, and moulded to the bottom of the register, as if it had been partially fused. It was hard and brittle. The vapour of the mercury had probably combined with it at some temperature below the boiling point; the amalgam so formed had flowed down to the bottom of the bar, and the mercury was afterwards expelled by the boiling temperature.

I may here observe, as not unworthy of attention, that in no instance have I seen a metal acted upon by the vapour of mercury at its full boiling temperature;—even gold, which has so strong an affinity for it, comes out of it with its yellow colour perfectly unstained; but when the mercury is in the fluid form at the same temperature, the gold is immediately dissolved by it.

Under these circumstances there certainly may exist some doubt whether the full amount of expansion in zinc to the boiling point of mercury was properly registered.

On the other hand, in confirmation of the result so recorded, it may be seen, in Table XIV. of the expansion of the alloys, that a composition of half copper and half zinc presents the same anomaly; the expansion for the 600° to boiling mercury is not quite four times that of the 150° to boiling water. In the alloy of three fourths copper to one fourth zinc, the rate of expansion increases in a small degree; and in common brass, where the proportion of zinc is still less, it increases still more rapidly.

My purpose in instituting these experiments upon the alloys, was to observe the relation which might exist between the expansions of the pure metals and those of their mixtures: and the better to illustrate any such, I made alloys of copper with known multiple proportions of zinc and tin. I shall here present, in a tabular form, the temperatures of their melting points, as derived from their expansions to the boiling points of water and mercury; as, although I am not able to compare them with results directly obtained by immersion, we can judge, by comparison with the similar calculation of the pure metals, within what limits any error is probably confined.

## Table XVI.

Fusing points of alloys, derived from their expansions to 212° and 662° supposed equable.

| | From 212° rate. | From 662° rate. |
|---|---|---|
| | ° | ° |
| Brass. Copper ½, Zinc ½ ..... | 1842 | 1750 |
| Brass. Copper ⅓, Zinc ⅔ ..... | 1672 | 1910 |
| Bronze. Copper ⅛, Tin ⅞ ..... | 1761 | 1690 |
| Bronze. Copper ⅓, Tin ⅔ ..... | 1773 | 1534 |
| Bronze. Copper ½, Tin ½ ..... | 1755 | 1446 |
| Pewter. Lead ⅔, Tin ⅓ ..... | 403 | |
| Type Metal. Lead and Antimony .... | 507 | |

I have not included in the foregoing Table the alloy of half copper and half tin, but have exhibited its expansion to the boiling point of mercury in Table XIV. This mixture was very hard and brittle, and resembled the speculum metal of reflecting telescopes. After it had been exposed to boiling mercury, it appeared as if it had undergone partial fusion; it was set fast in the cavity of the register, and had thickened towards the lower extremity. I am inclined to think that it had nearly attained its melting point, but it was broken in removing it; and I had not an opportunity of trying any further experiment with it.

With regard to these alloys, the experiments are not numerous enough to enable us to deduce with precision the general laws by which their expansions and points of fusion are governed; but enough is discernible to show that the subject is well worthy of further investigation. It appears

1st. That the expansion of the compounds is not the mean of the expansions of the simple metals of which they are composed, but bears some proportion to their relative quantities. Thus we may observe that the expansion of brass increases with the quantity of zinc which it contains, as does bronze or bell-metal with the quantity of tin.

2ndly. That the expansion of brass is in an increasing ratio to the increase of temperature till the quantity of zinc amounts to one half, when it seems to assume a decreasing rate, as we have reason to suppose is the case with pure zinc. On this account the melting points both of this mixture and zinc appear

to be higher when derived from their expansions to the boiling point of mercury, than when calculated from their expansions to the boiling points of water. With this exception, there is great reason to suppose that the melting points of the alloys, from the higher rate of expansion, cannot be very far removed from the true temperatures.

3rdly. That the melting point of copper is reduced by an admixture of one fourth of zinc to nearly the average which results from the proportions of the two ingredients; but by an admixture of an equal quantity of tin it is reduced in a much greater proportion. The temperature derived from the average with zinc would be 1690°, and the corresponding temperature in the Table is 1750°. The temperature derived from the average with tin would be 1607°, but the corresponding temperature is only 1446°.

4thly. That a similar power in tin to depress the melting point of another metal is exhibited in pewter; in which we may observe that a mixture of one fifth of tin with lead reduces the melting point actually below that of either of the pure metals; and we may recall to recollection the fact, that an alloy of eight parts of bismuth, whose fusing point is 476°; five of lead, whose fusing point is 612°; and three of tin, whose fusing point is 442°,—liquefies at 212°.

I shall here subjoin a Table, in the usual form, of the progressive linear dilatation by heat of such solids as I have measured with the pyrometer to the boiling point of water, the boiling point of mercury, and their respective melting points, where they have been ascertained. I have added to their apparent expansions by the register the corresponding expansion of the black-lead; upon the assumption that the latter continues at an equal rate to temperatures above 662°; in which it is not probable, from the preceding observations, that there is any error of material importance.

## Table XVII.

### Linear Dilatations of Solids by Heat.

Dimensions which a bar takes whose length at 62° is 1.000000.

| | At 212° (150°). | At 662° (600°). | At Point of Fusion. |
|---|---|---|---|
| Black-lead ware .... | 1.000244 | 1.000703 | |
| Wedgwood ware .... | 1.000735 | 1.002995 | |
| Platinum ..... | 1.000735 | 1.002995 | (1.009926 maximum, but not fused.) |
| Iron (wrought) ..... | 1.000984 | 1.004483 | (1.018378 to the fusing point of cast iron.) |
| Iron (cast)..... | 1.000893 | 1.003943 | 1.016389 |
| Gold ..... | 1.001025 | 1.004238 | |
| Copper ..... | 1.001430 | 1.006347 | 1.024376 |
| Silver..... | 1.001626 | 1.006886 | 1.020640 |
| Zinc ..... | 1.002480 | 1.008527 | 1.012621 |
| Lead ..... | 1.002323 | ..... | 1.009072 |
| Tin..... | 1.001472 | ..... | 1.003798 |
| Brass. Zinc 1 ..... | 1.001787 | 1.007207 | 1.021841 |
| Bronze. Tin 1..... | 1.001541 | 1.007053 | 1.016336 |
| Pewter. Tin 1..... | 1.001696 | ..... | 1.003776 |
| Type Metal ..... | 1.001696 | ..... | 1.004830 |

The regularity of these several expansions is very striking. As long as the metal retains the solid form, the dilatation proceeds according to a fixed law, without any sudden starts or changes; till assuming the form of a liquid it doubtless is subject to a different mode of action.

I shall conclude these observations with the results of some experiments which I made to determine, if possible, the cause of the singular change of texture in platinum, when intensely heated in the black-lead registers, which I described in my former paper. Upon showing the bar so changed to those who were best acquainted with the working of this metal, they universally ascribed it to the action of sulphur: but nobody could explain to me why this action should require such a very intense heat; as up to the temperature of melting cast iron, to which it had several times been exposed, no change took place; but the bar remained perfectly soft and malleable.

In De Ferussac's Bulletin for November 1830, there is an abstract of my paper on the Pyrometer, which the Editor concludes with the observation, that "unfortunately I inclosed in the crucible which contained the register

and the bar of platinum some pieces of iron, without being aware of the fact, which is known to all the workmen who manufacture platinum, that the mere presence of iron is enough to communicate brittleness to that metal."

Upon inquiry amongst workmen in this country I cannot find that such a property has ever been observed in the course of their experience; and when I consider that the bar in the cavity of the register was perfectly preserved from contact with the iron nails; and moreover, that it had actually been plunged into melted iron without any change of properties; I cannot suppose that the alteration depended in any way upon this circumstance.

To resolve these doubts I took 116 grains of the brittle platinum, which had been ground without difficulty to a fine powder in a steel mortar, and boiled them in nitro-muriatic acid till I had effected a complete solution;—a little of this solution produced a scarcely perceptible cloudiness in a solution of muriate of baryta. This I have reason to think was owing to a slight impurity in the acids employed; I infer therefore that there was no sulphur in the metal. I proceeded to evaporate the solution; which towards the end of the process assumed a gelatinous appearance. When in this state, I poured alcohol upon it; and as the acid still remained in excess, a violent reaction took place with extrication of nitrous gas. I then evaporated to dryness and continued the heat; till the salt of platinum kindled spontaneously, and finally was left in a spongy state. This was again digested in nitro-muriatic acid, and the solution carefully evaporated to dryness. The muriate of platinum was then dissolved in water, and a sandy residue remained; which, when well washed and heated to redness, was of a grayish-white colour, and had all the properties of silica: it weighed 3.5 grains. There can therefore, I think, be little doubt that at the high temperature to which it was exposed, platinum took up as much as 3 per cent of silica; or, more probably, a quantity of its base equivalent to that quantity of the earth, to which it owed all its change of character and properties. A temperature considerably above that of melting cast iron appears to be necessary to this combination; which is analogous in many respects to the absorption of carbon by iron in the process of making steel by cementation.

*Errata in the former Paper, Phil. Trans.* 1830.

The following errata in my former paper have arisen from my having omitted to add the initial 32° in the reduction of the Centigrade scale to that of FAHRENHEIT.

Page 266, line 24, *for* 360°, *read* 392°.
—— 266, — 26, — 360°, — 392°.
—— 266, — 27, — 360°, — 392°.
—— 269, — 3, — 360°, — 392°.
—— 269, — 5, — 360°, — 392°.
—— 269, — 23, — 360°, — 392°.
—— 269, — 25, — 360°, — 392°.

## XXV. *Experiments on the Length of the Seconds Pendulum at the Royal Observatory of Greenwich. By Captain* Edward Sabine, *of the Royal Regiment of Artillery, F.R.S.*

Read June 16, 1831.

THESE experiments were made with the original convertible pendulum constructed by Captain Kater, and employed by him in 1817 in Portland Place, London.

Prior to its employment in the present experiments I made the following alterations in the pendulum.

1. The tail pieces were removed altogether; coincidences being observed by the bar itself (the ends being blackened for the purpose), and by a disc of corresponding diameter on the pendulum of the clock.

2. The "moveable weight" employed by Captain Kater was dispensed with altogether; and the pendulum rendered convertible, within the limits of more exact adjustment by the slider, by filing away a small quantity of metal from one extremity of the bar. By this alteration there remained nothing moveable about the pendulum except the slider; and that was so placed in these experiments, that a change in its position, of so great magnitude as the tenth of an inch, did not occasion an alteration of so much as the tenth of a second in the daily rate of the pendulum, when suspended with the weight below, which is the position in which the rate is determined. The slider was clamped to the bar, and was moved by a screw for slow motion, by which it could be adjusted with tolerable precision to the hundredth of an inch. The graduation on the bar, by which the place of the slider was regulated, was on the side of the bar next the observer, who could thus at all times assure himself that no change occurred in its position by the inversion of the pendulum.

Neither of these alterations interfered with the distance between the knife edges, which was referred in 1817 to Sir George Shuckburgh's scale by Cap-

tain KATER, and found to measure 39.44085 inches of that scale at the temperature of 62°.

The experiments were made, unless where it is otherwise noticed, in the vacuum apparatus, established in the S.W. angle of the pendulum room in the Royal Observatory, being the place assigned for that apparatus by the Astronomer Royal.

The thermometer, by which the temperature of the pendulum was observed, is the same which I have used and described on former occasions; particularly in the Phil. Trans. for 1830, where its comparison is given with a standard thermometer of M. BESSEL's, at the same temperatures at which it was used in these experiments. In a paper on the construction and use of the vacuum apparatus, in the Phil. Trans. for 1829, I have shown that when the pressure of the air is withdrawn from the exterior of the bulb of this thermometer, an index correction of $+0°.75$ is required to make its indications in a highly rarified medium correspond with its measure of the same temperatures when under the pressure of the atmosphere. This index correction is consequently applied whenever the air is withdrawn from the apparatus. The thermometer was inclosed with the pendulum within the glasses, with its bulb suspended midway between the knife edges. When the air is withdrawn from the apparatus, and the shutters of the apartment are kept closed (except when light is required for the observation of coincidences), a great uniformity as well as steadiness of temperature is maintained within the glasses.

Whenever the indications of the barometer were required, a reference was made to the standard barometer of the observatory, which is stationed in an adjoining room on the same level.

The scale by which the arcs of vibration were observed was graduated in degrees, each degree being the 0.833th of an inch. In making the observation, the division of the scale coinciding with one side of the bar was noticed at each extremity of the vibration, and the same repeated with the other side of the bar; a mean was then taken of the two included spaces, and half the mean registered as the arc on either side the vertical. The scale being placed in both positions of the pendulum 45.5 inches below the point of suspension, the registered arcs multiplied by 1.05 produce the true arcs of vibration; by employing these in the usual formula for that purpose, what is usually considered

as the compensation for the arcs is obtained. During the progress of the experiments, however, I had occasion to suspect that, particularly when the pendulum was vibrated with the weight above, the retardation of the vibrations, in consequence of their being performed in circular arcs, was greater than the compensations computed by the customary formula $N \cdot \frac{\sin(A+a) \cdot \sin(A-a)}{32\,M \cdot (\log. \sin A - \log. \sin a)}$. To examine this more closely, and to obtain practically a just compensation for the arcs in which the pendulum had been vibrated in the course of its employment, I made several series of experiments distinct from those made to determine the rate, and which I shall proceed to describe in the first instance, though they were not the first in the order of time.

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*Correction for the Arcs of Vibration.*

§ 1. Weight above.

On the 27th of January 1830, I made the following observations, purposing to compare, under circumstances otherwise similar, the rate of the pendulum in different arcs; 1st, commencing with 1°.32 and ending with 0°.73; 2nd, commencing with 0°.70 and ending with 0°.42; and 3rd, commencing with 0°.42 and ending with 0°.19. Having withdrawn the air from the apparatus, the resistance to the vibration was so far diminished, that the time which the pendulum took to reduce its arc from 1°.32 to 0°.73, from 0°.70 to 0°.42, and from 0°.42 to 0°.19, was in each case sufficient to give the rate of the pendulum within the respective arcs with tolerable exactness.

The rate of the clock on this day was taken from its average rate in several days, viz. from the 24th to the 30th of January; exactness in the daily gain or loss of the clock was not required, as the observations were only to be used in their relation to each other.

January 27, 1830. Clock by Dent making 86311.0 Vibrations in a mean solar day.

| No. of Coincid. | Therm. | Gauge. | Times of Disapp. | Times of Reapp. | Times of Coincidence. | Arcs of Vibration. | Mean Temp. | Mean Gauge. | Mean Interval. | Reductions to mean tem. 36°. | Reductions to a Vacuum. | Vibrations per diem. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | in. | | | | |
| 1 | ..... | ..... | 11 16 | 11 27 | 10 11 21.5 | | | | | | | |
| 2 | 34.9 | 0.86 | 23 31 | 23 43 | 10 23 37 | 1.26 × 1.05 = 1.32 | 35.1 | | | | | |
| 3 | ..... | ..... | 35 45 | 35 59 | 10 35 52 | | + 0.75 | 1.275 | 737.90 | −0.07 | +0.60 | 86077.59 |
| 15 | 35.3 | 1.69 | 3 15 | 3 44 | 1 3 29.5 | 0.70 × 1.05 = 0.73 | 35.85 | | | | | |
| | Air pumped out to reduce the gauge. | | | | | | | | | | | |
| | | | | | | | 35.35 | | | | | |
| 16 | 35.3 | 0.92 | 15 35 | 16 02 | 1 15 48.5 | 0.67 × 1.05 = 0.70 | + 0.75 | | | | | |
| 31 | 35.4 | 1.53 | 21 06 | 21 47 | 4 21 26.5 | 0.40 × 1.05 = 0.42 | 36.10 | 1.225 | 742.53 | +0.04 | +0.58 | 86079.14 |
| 31 | 35.4 | 1.53 | 21 06 | 21 47 | 4 21 26.5 | 0.40 × 1.05 = 0.42 | 35.45 | | | | | |
| 46 | ..... | ..... | 26 37 | 28 03 | 7 27 20 | | + 0.75 | | | | | |
| 47 | 35.5 | 1.92 | 38 55 | 40 26 | 7 39 40.5 | 0.18 × 1.05 = 0.19 | | 1.725 | 743.41 | +0.08 | +0.82 | 86079.66 |
| 48 | ..... | ..... | 51 17 | 52 48 | 7 52 02.5 | | 36.20 | | | | | |

Examining these observations we have

| | I. | II. | III. |
|---|---|---|---|
| Arcs . . . . . | 1.32 to 0.73 | 0.70 to 0.42 | 0.42 to 0.19 |
| Vibrations per diem, uncorrected for the arcs . . . . | 86077.59 | 86079.14 | 86079.66 |
| Compensation for the arcs by the formula . . . . . | +1.67 | +0.50 | +0.14 |
| These numbers should agree if the corrections for the arcs computed by the formula were a just compensation . | 86079.26 | 86079.64 | 86079.80 |

From the differences that are here seen to exist, it appears that when the pendulum is vibrated with the weight above, the compensation for the arcs as computed by the customary formula is in defect; and more in defect as the arcs are larger. To make the results agree, it is necessary to multiply the computed correction by 1.36. By the employment of this multiplier, the result obtained in the arcs of least magnitude (which as being the nearest to infinitely small arcs are presumed to be the most correct,) is not altered more than five-hundredths of a vibration; whilst the results obtained in larger arcs

are brought in accordance with it. The corrections and the corrected vibrations then become

| | I. | II. | III. |
|---|---|---|---|
| Corrections for the arcs . . . . | +2.27 | +0.68 | +0.19 |
| | 86077.59 | 86079.14 | 86079.66 |
| Corrected vibrations . . . . | 86079.86 | 86079.82 | 86079.85 |

On the 29th January the same process was repeated in arcs differing somewhat from the preceding, and designed therefore to render the differences in the results, should they be found to exist, still more conspicuous.

| No. of Coincid. | Therm. | Gauge. | Times of Disapp. | Times of Reapp. | Times of Coincidence. | Arcs. | Mean Temp. | Mean Gauge. | Mean Interval. | Reductions to 36°. | Reductions to a Vacuum. | Vibrations per diem. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° ′ ″ | ° | in. | | | | |
| 1 | 34.6 | 0.80 | 17 53 | 18 01 | 1 17 57 | 1.39×1.05=1.46 | 34.55 | | | | | |
| 17 | 34.5 | 1.05 | 34 26 | 34 54 | 4 34 40 | 0.76×1.05=0.80 | + 0.75 | 0.925 | 737.69 | −0.31 | +0.42 | 86077.11 |
| | | | | | | | 35.30 | | | | | |
| 35 | 34.4 | 1.23 | 17 20 | 18 04 | 8 17 42 | 0.39×1.05=0.41 | 34.45 + 0.75 = 35.20 | 1.14 | 743.44 | −0.35 | +0.54 | 86078.98 |
| 53 | 34.1 | 1.36 | 0 39 | 02 04 | 12 01 21.5 | 0.17×1.05=0.18 | 34.25 + 0.75 = 35.00 | 1.295 | 745.53 | −0.44 | +0.61 | 86079.60 |

Examining these observations, we have

| | I. | II. | III. |
|---|---|---|---|
| Arcs . . . . . | 1.46 to 0.80 | 0.80 to 0.41 | 0.41 to 0.18 |
| Vibrations, uncorrected for the arcs . . . . . | 76077.11 | 76078.98 | 76079.60 |
| Corrections for the arcs by the formula . . . . . | +2.03 | +0.58 | +0.13 |
| Numbers which should agree if the corrections computed by the formula were a just compensation . . . . . | 76079.14 | 76079.56 | 76079.73 |

The inference from this day's experiment is to the same effect as before. The multiplier required in this instance is 1.32, altering the result in the smallest arcs four hundredths of a vibration, and bringing the others in accordance with it. The corrections and the corrected vibrations become

| | I. | II. | III. |
|---|---|---|---|
| Corrections . . . . . | +2.68 | +0.77 | +0.17 |
| | 86077.11 | 86078.98 | 86079.60 |
| Corrected vibrations . . . | 86079.79 | 86079.75 | 86079.77 |

On February 3rd I made a third series with the pendulum in the same position; the pendulum of the clock had been shortened since January 29th, and was now making about 86371.0 vibrations; but the temperature being at this time very low, its going was less regular than usual. To obviate that inconvenience I commenced with a series in large arcs; made then two series in small arcs, and concluded with a second series in large arcs; and finally took a mean between the two series in large arcs and the two series in small arcs, as the results to be compared with each other.

| No. of Coincid. | Therm. | Gauge. | Times of Disapp. | Times of Reapp. | Times of Coincidence. | Arcs. | Mean Temp. | Mean Gauge. | Mean Interval. | Reductions to 25°. | Reductions to a Vacuum. | Vibrations per diem. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | δ ° | ° | in. | | | | |
| 1 | 25.0 | 1.30 | 20 07 | 20 15 | 2 20 11 | 1.26×1.05=1.32 | 25.0 | | | | | |
| 13 | 25.0 | 1.30 | 18 40 | 18 55 | 4 18 47.5 | 0.85×1.05=0.89 | + 0.75 | 1.30 | 593.04 | +0.33 | +0.64 | 86080.67 |
| | | | | | | | 25.75 | | | | | |
| | Fresh impulse. | | | | | | | | | | | |
| 1 | 24.9 | 1.30 | 26 23 | 26 55 | 4 26 39 | 0.35×1.05=0.37 | 24.8 | | | | | |
| 13 | 24.7 | 1.30 | 25 51 | 26 46 | 6 26 18.5 | 0.25×1.05=0.26 | + 0.75 | 1.30 | 598.29 | +0.24 | +0.64 | 86083.16 |
| | | | | | | | 25.55 | | | | | |
| 26 | 24.7 | 1.34 | 35 25 | 36 35 | 8 36 00 | 0.15×1.05=0.16 | 24.70 + 0.75 = 25.45 | 1.32 | 598.58 | +0.20 | +0.65 | 86083.27 |
| | Fresh impulse. | | | | | | | | | | | |
| 1 | 24.8 | 1.34 | 04 03 | 04 09 | 9 04 06 | 1.37×1.05=1.44 | 24.90 | | | | | |
| 16 | 25.0 | 1.40 | 31 41 | 31 57 | 11 31 49 | 0.80×1.05=0.84 | + 0.75 | 1.37 | 590.87 | +0.29 | +0.67 | 86079.58 |
| | | | | | | | 25.65 | | | | | |

Examining these observations, we have

| | I. | IV. | II. | III. |
|---|---|---|---|---|
| Arcs . . . . . | 1°.32 to 0°.89 | 1°.44 to 0°.84 | 0°.37 to 0°.26 | 0°.26 to 0°.16 |
| Vibrations . . . . | 86080.67 | 86079.58 | 86083.16 | 86083.27 |
| Correction for the arcs by the formula . . | +1.98 | +2.08 | +0.16 | +0.06 |
| Numbers which should agree if the corrections computed by the formula were a just compensation . . . . | 86082.15 | | 86083.32 | |

The inference here is again to the same effect. The multiplier required to produce accordance is 1.61. The corrections and the corrected vibrations then become

| | I. & IV. | II. & III. |
|---|---|---|
| Corrections . . . . . | +3.27 | +0.18 |
| Corrected vibrations . . . . . | 86083.39 | 86083.39 |

For a purpose foreign to the present object, the slider was on this day at 1.75 inch from the middle of the pendulum towards the weight.

The irregular going of the clock (which stopped altogether soon after the fourth series was completed) may have rendered the result of this day's experiments of less value than that of either of the preceding days, in determining the amount of the multiplier. It was not until the 5th of May following, that, the clock being repaired and replaced, I was able to resume the inquiry by the following observations conducted in the same manner as before. The clock was making 86665.0 vibrations, on an average of several days at this period.

| No. of Coincid. | Therm. | Gauge. | Times of Disapp. | Times of Reapp. | Times of Coincidence. | Arcs. | Mean Temp. | Mean Gauge. | Mean Interval. | Reductions to 58°. | Reductions to a Vacuum. | Vibrations per diem. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° ° | ° | in. | | | | |
| 1 | .... | .... | 53 31 | 53 33 | 2 53 32 | | | | | | | |
| 2 | 56.5 | 2.1 | 58 21 | 58 25 | 2 58 23 | 1.36×1.05=1.43 | 56.60 | | | | | |
| 3 | .... | .... | 03 13 | 03 15 | 3 03 14 | | + 0.75 | 2.10 | 291.035 | −0.29 | +0.96 | 86065.61 |
| 20 | .... | .... | 25 39 | 25 43 | 4 25 41 | | | | | | | |
| 21 | 56.7 | 2.1 | 30 31 | 30 35 | 4 30 33 | 0.93×1.05=0.98 | 57.35 | | | | | |
| 22 | .... | .... | 35 23 | 35 25 | 4 35 24 | | | | | | | |
| | Fresh impulse. | | | | | | | | | | | |
| 1 | 57.0 | .... | 47 36 | 47 43 | 4 47 39.5 | | | | | | | |
| 2 | .... | 2.1 | 52 29 | 52 35 | 4 52 32 | 0.24×1.05=0.26 | 57.32 | | | | | |
| 3 | 57.1 | .... | 57 21 | 57 27 | 4 57 24 | | + 0.75 | 2.10 | 292.46 | +0.03 | +0.96 | 86068.85 |
| 21 | 57.6 | .... | 25 06 | 25 13 | 6 25 09.5 | | | | | | | |
| 22 | .... | 2.1 | 30 56 | 31 05 | 6 30 00.5 | 0.18×1.05=0.19 | 58.07 | | | | | |
| 23 | 57.6 | .... | 34 49 | 34 57 | 6 34 53 | | | | | | | |

Examining these observations, we have

| | I. | II. |
|---|---|---|
| Arcs | 1.43 to 0.98 | 0.26 to 0.19 |
| Vibrations | 86065.61 | 86068.85 |
| Correction for the arcs by the formula | +2.35 | +0.08 |
| Numbers which should agree if the corrections computed by the formula were a just compensation. | 86067.96 | 86068.93 |

The inference is the same as on the former occasions. The multiplier is in this instance 1.42, and the corrections and corrected vibrations become

| | | |
|---|---|---|
| Corrections | +3.34 | +0.11 |
| | 86065.61 | 86068.85 |
| Corrected vibrations | 86068.95 | 86068.96 |

If now we collect in one view the different multipliers which have been found from these four series of experiments, we have January 27, 1.36; January 29, 1.32; February 3, 1.61; May 5, 1.42: of which four results, that of February 3rd (1.61), from the unfavourable circumstances of the experiments in the irregular going of the clock already noticed, is least entitled to confidence. The arithmetical mean of the four is 1.43; and that of the three, omitting the result of February 3rd, is 1.37. If then we take 1.4 as

the multiplier, to be employed in computing the compensations for the arcs in the experiments with the weight above, to be hereafter related, we may infer with probability that we employ a more correct multiplier than either 1.3 or 1.5. Now, if either of those numbers were substituted for 1.4, the effect would be to alter the rate of the pendulum with the weight above, as deduced from the experiments in the succeeding pages, one tenth of a vibration per diem. To correct the influence of this tenth of a vibration on the convertibility (the object for which the pendulum is vibrated with the weight above), the slider would require to be moved a certain quantity, which would alter the final deduction of the rate, due to the distance between the axes of suspension, less than one hundredth of a second per diem. In employing 1.4 therefore as a multiplier, any uncertainty in the final result, arising from this cause, is limited in all probability to less than .01 of a second per diem.

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## § 2. Weight below.

On the 31st of January and 1st of February 1830, I made the following observations, for the purpose of ascertaining the compensation for the arcs when the weight was below. The clock, by DENT, was making 86313.3 vibrations. The clock had been stopped on the 30th of January, and was stopped again on the 1st of February, to alter its pendulum. The rate was assigned by comparison with the transit clock during the two days it was going.

| No. of Coincid. | Therm. | Gauge. | Times of Disapp. | Times of Resp. | Times of Coincidence. | Arcs. | Mean Temp. | Mean Gauge. | Mean Interval. | Reductions to 30°. | Reductions to a Vacuum. | Vibrations per diem. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° ° | ° | in. | | | | |
| 1 | .... | .... | 16 00 | 16 35 | 9 16 17.5 | 0.40 × 1.05 = 0.42 | 29.9 | | | | | |
| 2 | 30.3 | 1.20 | 28 19 | 28 55 | 9 28 37 | | + 0.75 | | | | | |
| 21 | 29.5 | 1.30 | 22 25 | 23 18 | 1 22 51.5 | 0.23 × 1.05 = 0.24 | 30.65 | 1.25 | 739.8 | +0.29 | +0.52 | 86080.77 |
| 22 | .... | .... | 34 49 | 35 41 | 1 35 15 | | | | | | | |
| | Fresh impulse. | | | | | | | | | | | |
| 1 | 29.5 | 1.30 | 49 28 | 49 36 | 1 49 32 | 1.25 × 1.05 = 1.31 | 28.75 | | | | | |
| 39 | 28.0 | 1.40 | 36 18 | 36 48 | 9 36 33 | 0.50 × 1.05 = 0.52 | + 0.75 | | | | | |
| | | | | | | | 29.50 | 1.35 | 737.4 | −0.22 | +0.56 | 86079.54 |
| 88 | 28.4 | 1.52 | 41 48 | 43 26 | 19 42 37 | 0.14 × 1.05 = 0.15 | 28.20 | | | | | |
| | | | | | | | + 0.75 | | | | | |
| | | | | | | | 28.95 | 1.46 | 742.12 | −0.46 | +0.61 | 86080.81 |

Examining these observations, we have

| | I. | II. | III. |
|---|---|---|---|
| Arcs | 0.42 to 0.24 | 1.31 to 0.52 | 0.52 to 0.15 |
| Vibrations | 86080.77 | 86079.54 | 86080.81 |
| Correction for the arcs by the formula | +0.18 | +1.29 | +0.17 |
| Numbers which should agree if the corrections computed by the formula were just compensations | 86080.95 | 86079.83 | 86080.98 |

On the 28th of April following, I repeated the experiment in arcs designed to make the differences between the experimental and computed results still more apparent. The clock, by DENT, was making 86660.5 vibrations on an average of several days about that period.

| No. of Coincid. | Therm. | Gauge. | Times of Disapp. | Times of Reapp. | Times of Coincidence. | Arcs. | Mean Temp. | Mean Gauge. | Mean Interval. | Reductions to 58°. | Reductions to a Vacuum. | Vibrations per diem. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° ° | ° | in. | | | | |
| 1 | 53.4 | 1.60 | 27 46 | 27 48 | 12 27 47 | 1.42×1.05=1.49 | 53.70 | | | | | |
| 13 | 54.0 | 1.61 | 26 05 | 26 07 | 1 26 06 | 1.25×1.05=1.31 | + 0.75 | 1.605 | 291.58 | −1.56 | +0.65 | 86065.17 |
| | | | | | | | 54.45 | | | | | |
| 26 | 55.4 | 1.62 | 29 18 | 29 21 | 2 29 19.5 | 1.07×1.05=1.12 | 54.7 + 0.75 = 55.45 | 1.615 | 291.81 | −1.12 | +0.67 | 86066.09 |
| | Fresh impulse. | | | | | | | | | | | |
| 1 | 56.0 | 1.62 | 51 49 | 51 52 | 2 51 51.5 | 0.38×1.05=0.40 | 56.2 | | | | | |
| 15 | 56.4 | 1.62 | 00 07 | 00 11 | 4 00 09 | 0.31×1.05=0.33 | + 0.75 | 1.62 | 292.68 | −0.46 | +0.67 | 86068.53 |
| | | | | | | | 56.95 | | | | | |
| 25 | 56.2 | 1.63 | 48 55 | 49 00 | 4 48 57.5 | 0.28×1.05=0.30 | 56.3 + 0.75 = 57.05 | 1.625 | 292.85 | −0.42 | +0.68 | 86068.92 |
| | Fresh impulse. | | | | | | | | | | | |
| 1 | 56.2 | 1.63 | 56 49 | 56 51 | 4 56 50 | 1.30×1.05=1.36 | 56.3 | | | | | |
| 15 | 56.4 | 1.64 | 04 49 | 04 52 | 6 04 50.5 | 1.12×1.05=1.17 | + 0.75 | 1.635 | 291.46 | −0.42 | +0.68 | 86066.10 |
| | | | | | | | 57.05 | | | | | |

Examining these observations, we have

| | I. | II. | III. | IV. | V. |
|---|---|---|---|---|---|
| Arcs . . . . . | 1.49 to 1.31 | 1.31 to 1.12 | 0.40 to 0.33 | 0.33 to 0.30 | 1.36 to 1.17 |
| Vibrations . . . . . | 86065.17 | 86066.09 | 86068.53 | 86068.92 | 86066.10 |
| Correction for the arcs by the formula . . . | +3.13 | +2.42 | +0.22 | +0.16 | +2.73 |
| Numbers which should agree if the corrections computed by the formula were just compensations | 86068.30 | 86068.51 | 86068.75 | 86069.08 | 86068.83 |

In both these series of experiments the same indication is afforded, viz. that the retardation of the vibration is greater in large arcs than is covered by the correction computed by the formula. The difference, however, between the results in large arcs and in small arcs, with the computed corrections applied, is much less than takes place when the pendulum is vibrated with the weight above. With the weight below, the computed corrections being multiplied by 1.13, the results are rendered accordant with each other on both days of experiment, and the vibrations become as follows:—

January 31st, large arcs, Exp. II. 86080.99; small arcs, Exp. I. & III. 86081.00.
April 28th, large arcs, Exp. I. II. & V. 86068.92; small arcs, Exp. III. & IV. 86068.94.

The alteration produced by the employment of this multiplier, in the final deduction of the rate of the pendulum from the experiments to be subsequently narrated, is an addition of 0.12 vibration per diem to the number of vibrations which would otherwise have been derived. I should have been glad to have employed a greater number of observations in the more assured determination of this multiplier, but circumstances did not permit me to pursue the inquiry further; and I have only now to refer to the consistency and concurrent indication of those results that were obtained, as an evidence that the multiplier derived from them is in all probability very near the truth. I may also notice, that a change of one unit in the second figure of decimals of this multiplier, would produce an alteration of something less than one hundredth of a second per diem in the rate of the pendulum, derived from the experiments which form the subject of this paper.

It was my intention to have investigated experimentally the cause of the retardation being greater in large arcs than accords with the formula of reduction

to infinitely small arcs; but circumstances not permitting me to do so at present, I have only to state my conjecture, that it is caused by the gliding of the knife edges on the planes, a consequence of the elasticity of the planes. It has been found by M. Bessel, that small movements of this kind always took place in a pendulum vibrating on a knife edge, whatever might be the nature of the supporting planes; that its direction was the same as the motion of the pendulum,—to the right when the pendulum moved to the right, and vice versâ; and that its amount was proportioned to the arcs of vibration. Supposing that the cause is as I have conjectured, it would have been satisfactory to have measured the amount of the gliding corresponding to particular arcs, *directly*, in the manner that M. Bessel has done, and to have compared the correction of the *length*, which is the mode of compensation adopted by M. Bessel, with the correction of the *rate*, which is the method that has been adopted here; by either mode the experiments ought to give the same length for the seconds pendulum.

In discussing the correction for the arcs of vibration, it has appeared the most satisfactory course, to introduce the detail of the experiments relating to that branch of the subject into the body of the discussion itself. In the remainder of the paper I shall pursue the more usual course, of placing together, at the close, the detail of all the experiments, in the order and succession in which they were made; introducing into the discussion, abstracts and results, with proper references to the part of the paper containing the details.

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Before we enter on the examination of the rate of the pendulum, it is necessary to ascertain the reduction to a vacuum for the small residue of air which cannot be pumped out of the apparatus, as well as for the small additional quantity which occasionally leaks in. I proceed to collect in one view the results of the experiments which were made at suitable opportunities to determine the amount of this reduction.

### *Reduction to a Vacuum.*

§ 1. Weight above.

The plan of experiment was to compare the vibrations made in full atmospheric pressure, with those in the exhausted apparatus, in circumstances in

other respects as nearly similar as possible. By this comparison, the correction for nearly the full atmospheric pressure is ascertained; the proportional part of which is taken as the equivalent for the small residue of air which cannot be wholly withdrawn, and for the occasional small leakage of the apparatus during the experiments; the pressure withinside the glasses being duly observed and registered by means of a syphon gauge placed by the pendulum. In regarding the correction for the last remaining inch of pressure as a proportional part, namely the thirtieth, of the correction for the whole atmosphere, I refer to the experiments related in a former paper in the Philosophical Transactions, in which it was shown that the corrections for a half atmosphere, and for a quarter atmosphere, bore corresponding proportions to the correction for the full atmosphere.

The following is an abstract of two series of such experiments with the weight above.

| | Reference to the details at the close of the paper. | Therm. | Vibrations. | Reduction to mean temp. 57°. | Vibrations at 57°. | | Barom. or Gauge. | |
|---|---|---|---|---|---|---|---|---|
| | | ° | | ° | | | inch. | |
| In the free air . | E | 57.5 | 86056.88 | +0.22 | 86057.10 | 86057.03 | 29.650 | 29.433 |
| | G | 59.6 | 86055.82 | +1.14 | 86056.96 | | 29.217 | |
| Air withdrawn . | F | 59.8 | 86068.56 | +1.23 | | 86069.79 | | 1.045 |
| (Slider at 1.5) | | | | Differences ..... | | 12.76 | | 28.388 |

| | | | | | | | inch. |
|---|---|---|---|---|---|---|---|
| In the free air . | O | 56.05 | 86055.95 | −0.41 | | 86055.54 | 29.376 |
| Air withdrawn . | P to S (mean) | 56.87 | 86068.48 | −0.06 | | 86068.42 | 1.572 |
| (Slider at 1.633) | | | | Differences ..... | | 12.80 | 27.804 |

We have then, from these two series, the following differences:

12.76 vibr. corresponding to 28.388 in. of air at 58.5; or 13.48 vibr. to 30 in.
12.88 —— —— 27.804 —— at 56.1; or 13.89. ——

Whence we obtain the reduction, when the weight is above, in the proportion of 13.68 vibrations per diem for 30 inches of air at 57°.3.

## § 2. Weight below.

Abstract of three series of experiments from which the reduction to a vacuum is derived for the vibrations of the pendulum with the weight below.

| | Reference to the details. | Temp. | Vibrations. | Reduction to mean temp. 57°. | Vibrations at 57°. | | Barom. or Gauge. | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | inch. | |
| In free air .... | A | 56.95 | 86057.30 | −0.02 | 86057.28 | 86057.25 | 29.727 | 29.740 |
| | B | 57.60 | 86056.88 | +0.26 | 86057.14 | | 29.756 | |
| | D | 56.70 | 86057.48 | −0.13 | 86057.35 | | 29.737 | |
| Air withdrawn . | C | 56.95 | 86068.28 | −0.02 | | 86068.24 | | 1.950 |
| (Slider at 1.5) | | | Differences ..... | | | 10.99 | | 27.790 |

| | | | | | | |
|---|---|---|---|---|---|---|
| In free air .... | N | 57.50 | 86056.84 | +0.22 | 86057.06 | 29.457 |
| Air withdrawn . | H to M | 56.87 | 86068.67 | −0.06 | 86068.61 | 1.211 |
| (Slider at 1.633) | | | Differences ..... | | 11.55 | 28.246 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| In free air .... | FF | 57.50 | 86056.88 | +0.22 | 86057.10 | 86057.04 | 29.361 | 29.349 |
| | GG | 58.00 | 86056.53 | +0.44 | 86056.97 | | 29.337 | |
| Air withdrawn . | W to EE | 56.75 | 86068.71 | −0.11 | | 86068.60 | | 0.976 |
| (Slider at 1.566) | | | Differences ..... | | | 11.56 | | 28.373 |

From these three series we have the following differences; viz.

10.99 vibr. corresponding to 27.790 in. of air at 57°.1; or 11.86 vibr. to 30 in.
11.55 —— —— 28.246 —— at 57.5; or 12.26 ——
11.56 —— —— 28.373 —— at 57.7; or 12.22 ——

Whence we obtain the reduction when the weight is below, in the proportion of 12.10 vibrations per diem for 30 inches of air at 57°.4.

The mean pressure within the glasses during the fourteen experiments with the weight below, (from which, as will be seen hereafter, the rate of the pendulum was finally derived,) was 1.07 inch: the reduction for which, according to the proportion found above, is 0.43 of a vibration per diem. This amount would be altered only one hundredth of a vibration, were the partial result most distant from the mean substituted for the mean of the three results by which the reduction was determined.

*Abstract of the Experiments for the Rate of the Pendulum.*

| Reference. | Gauge. | Temp. | Duration of experiments. | Vibrations. | Correction to M. T. 57°. | Reduction to a Vacuum. | Corrected Vibrations at 57° in a vacuum. |
|---|---|---|---|---|---|---|---|
| | inches. | ° | h m s | | | | |
| **I. Slider at 1.633. *a.* Weight below.** | | | | | | | |
| H | 1.207 | 57.37 | 13 58 29 | 86068.49 | +0.16 | +0.48 | 86069.13 |
| I | 0.930 | 56.61 | 9 17 09.5 | 86069.04 | −0.17 | +0.37 | 86069.24 |
| K | 1.365 | 56.25 | 14 35 26.5 | 86068.80 | −0.33 | +0.55 | 86069.02 |
| L | 0.938 | 56.67 | 8 48 33 | 86069.01 | −0.14 | +0.37 | 86069.24 |
| M | 1.615 | 57.47 | 14 05 25 | 86067.99 | +0.21 | +0.65 | 86068.85 |
| | 1.211 | 56.87 | 60 15 03 | 86068.67 | −0.06 | +0.48 | 86069.10 |
| **I. Slider at 1.633. *b.* Weight above.** | | | | | | | |
| P | 1.442 | 56.70 | 4 55 48 | 86068.67 | −0.13 | +0.66 | 86069.20 |
| Q | 1.650 | 56.70 | 4 41 01 | 86068.45 | −0.13 | +0.75 | 86069.07 |
| R | 1.685 | 56.55 | 8 32 43.5 | 86068.45 | −0.20 | +0.76 | 86069.01 |
| S | 1.512 | 57.55 | 6 15 10 | 86068.34 | +0.24 | +0.69 | 86069.27 |
| | 1.572 | 56.87 | 24 24 42.5 | 86068.48 | −0.06 | +0.72 | 86069.14 |
| **II. Slider at 1.566. *b.* Weight above.** | | | | | | | |
| T | 1.222 | 58.00 | 5 46 36.5 | 86068.76 | +0.44 | +0.56 | 86069.76 |
| U | 1.460 | 57.00 | 9 45 39 | 86068.83 | 0.00 | +0.67 | 86069.50 |
| V | 1.115 | 56.60 | 6 44 40 | 86069.24 | −0.17 | +0.51 | 86069.58 |
| | 1.266 | 57.20 | 22 16 55.5 | 86068.94 | +0.09 | +0.58 | 86069.61 |
| **II. Slider at 1.566. *a.* Weight below.** | | | | | | | |
| W | 0.950 | 56.85 | 15 03 24 | 86068.72 | −0.07 | +0.38 | 86069.03 |
| X | 1.055 | 56.40 | 13 01 03 | 86068.78 | −0.26 | +0.42 | 86068.94 |
| Y | 1.165 | 56.50 | 13 01 05.8 | 86068.82 | −0.22 | +0.46 | 86069.06 |
| Z | 0.985 | 56.95 | 10 21 25.7 | 86068.70 | −0.02 | +0.39 | 86069.07 |
| A A | 0.900 | 57.30 | 15 46 32 | 86068.56 | +0.13 | +0.36 | 86069.05 |
| B B | 0.895 | 57.60 | 9 01 09.5 | 86068.41 | +0.26 | +0.36 | 86069.03 |
| C C | 0.920 | 56.80 | 9 30 48.5 | 86068.67 | −0.09 | +0.37 | 86068.95 |
| D D | 1.070 | 56.50 | 9 23 49 | 86068.97 | −0.22 | +0.42 | 86069.17 |
| E E | 1.030 | 56.50 | 17 36 24 | 86068.89 | −0.22 | +0.42 | 86069.09 |
| | 1.000 | 56.82 | 112 45 41.5 | 86068.72 | −0.08 | +0.40 | 86069.04 |

The letters of reference in the first column are to the detail of the experiments at the close of the paper.

If, now, we take a mean of the results in both positions of the pendulum *a* and *b*, and in both positions of the slider I. & II., we have 86069.38 vibrations with the weight above, and 86069.07 vibrations with the weight below, corresponding to the slider at 1.6 inch (being the mean between 1.633 and 1.566) from the middle of the pendulum towards the weight.

The vibrations with the weight above are in excess 0.31 of a vibration. To ascertain the fraction of the tenth of an inch which it would be necessary to move the slider in order to produce perfect convertibility, and the effect of such change in the position of the slider, upon the vibrations with the weight below, we have from the experiments the following data (the letters referring as before to the detail of the experiments at the close of the paper).

Weight above.

| | | Vibrations. | |
|---|---|---|---|
| Slider at 1.5 | E to G | 86070.26 | making a difference of 1.12 vibration per diem corresponding to 0.133 inch of the slider. |
| ——— 1.566 | T to V | 86069.61 | |
| ——— 1.633 | P to S | 86069.14 | |

Weight below.

| | | Vibrations. | |
|---|---|---|---|
| Slider at 1.5 | A to D | 86069.00 | making a difference of 0.10 vibration per diem corresponding to 0.133 inch of the slider. |
| ——— 1.566 | W to E E | 86069.04 | |
| ——— 1.633 | H to M | 86069.10 | |

Whence we obtain by proportion 1.637 as the position of the slider which renders the pendulum perfectly convertible: and the vibrations, in the position of perfect convertibility, 86069.09 with the weight above, and 86069.10 with the weight below;—a result which, it will be seen, is in the closest accordance with the experiments H to S, during which the slider was at 1.633.

### *Expansion of the Pendulum.*

The rate of the pendulum having thus been found for the temperature of 57°, we have to seek its rate at 62°, being the temperature at which Captain Kater referred the distance between the knife edges to Sir George Shuckburgh's scale. To obtain the alteration of rate corresponding to each degree of Fahrenheit from the data furnished by the experiments, we have the rate with the weight below, observed on the 31st of January and 1st of February

at 30°, pp. 467 and 469, to compare with the rate at 57° which has been deduced as above from the experiments in the preceding August and September. The position of the slider in the winter experiments was at 1.6; consequently the result is strictly comparable with the mean of the two results H to M, and W to EE, made in summer, in one of which series the slider was at 1.633, and in the other at 1.566, the mean of which is 1.6. But in fact a small change of position of the slider when the weight is below, has so little influence on the rate, that had there been even a slight difference it might have been safely disregarded for the purpose now in view.

| | | | |
|---|---|---|---|
| Winter experiments . . . | 86081.00 | vibrations at | 30° |
| Summer experiments . . . | 86069.07 | —— | 57 |
| Differences . . . | 11.93 | —— | 27 |

Equivalent to 0.441 for each degree of FAHRENHEIT.

In the Phil. Trans. for 1830, Art. XIX. I have reported the results of a similar comparison of the rates of an invariable pendulum, made also of plate brass, in winter and in summer; by which it was shown that a degree of FAHRENHEIT corresponded to a change of 0.44 in the rate of that pendulum also. These two results are strongly confirmatory of each other.

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It results, then, that the vibrations of Captain KATER's pendulum, which at 57° have been found 86069.10, are 86066.90 at 62°. The distance between the knife edges as measured by Captain KATER is 39.44085 inches of Sir GEORGE SHUCKBURGH's scale, the pendulum and scale being at 62°. The vibrations in a vacuum corresponding to this distance are 86066.90, the temperature of the pendulum being 62°. We have then for the seconds pendulum in the Royal Observatory, at 62° and in a vacuum,

$$\text{In } 86066.90^{s} : 39.44085 \text{ In.} :: 86400^{s} : 39.13734 \text{ In.}$$

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*Stability of the support of the Pendulum in the preceding experiments.*

Being desirous of assuring myself of the stability of the support of the pendulum in the vacuum apparatus, I undertook a distinct series of experiments to obtain the rate of the pendulum on the iron frame, which is permanently affixed to the south wall of the pendulum room in the Royal Observatory, and designed for the use of observers with invariable pendulums, who wish to obtain a basis for their experiments on the variation of gravity at other stations. With this intention, I transferred the agate planes, which had been employed in the experiments in the vacuum apparatus, to the iron frame; and placed the pendulum on them, with the weight below, and the slider at 1.6 inch, making the observations H H to R R as detailed at the close of this paper. The experiments were necessarily made in the free air of the apartment, and are reduced to a vacuum by the reduction already found for the pendulum with the weight below. The rate of the clock, by GRAHAM, which is attached to the wall of the room beneath the iron frame, was furnished me by Mr. THOMAS GLANVILLE TAYLOR, in a memorandum which is subjoined to the detail of the observations. By a mean of the ten experiments H H to R R on the iron frame, the pendulum was found to make 86070.98 vibrations at 53° reduced to a vacuum. The equivalent at 57° is 86069.20, which differs by only 0.10 of a vibration per diem from 86069.10, the rate ascertained in the vacuum apparatus. The rate on the iron frame was obtained by thirty-three hours' vibration of the pendulum; that in the vacuum apparatus by 173 hours. So near an approximation, obtained in less than one fifth of the time that the experiments in the vacuum apparatus were continued, satisfied me that no permanent cause of difference existed, and that it only required that the experiments on the iron frame should be persevered in for the same length of time as those were in the vacuum apparatus, to produce the closest accordance. We may regard therefore the result of the experiments H H to R R, as establishing the equal stability of both supports; and as affording a fair inference that both supports are perfectly stable.

*Detail of the Experiments referred to in this paper.*

Preliminary Experiments to adjust the Slider, and to obtain the reduction to a vacuum for portions of air remaining in the Apparatus.—The registered arcs must be multiplied by 1.05 throughout, to give the true arcs.

Slider at 1.5 in. from the middle of the pendulum towards the weight. Weight below.

**Exp. A.** In full atmospheric pressure. Aug. 21, 1829 A.M. to P.M. Dent making 86465.92 vibr.

Barom. { beginning 29.766 / ending .. 29.796 } 29.781 at 57°; +.019 Capill.; −.073 to 32° = 29.727.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arcs. | Mean Temp. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.7 | .... | 50 42 | 50 58 | 9 50 50 | 0.65 | 56.95 | 422.95 | .... | +0.26 | 86057.30 |
| 33 | 57.2 | .... | 35 57 | 36 52 | 1 36 24.5 | 0.13 | | | | | |

**Exp. B.** In full atmospheric pressure. Aug. 21 P.M. Dent making 86465.92 vibrations.

Barom. { beginning 29.796 / ending .. 29.830 } 29.813 at 58°; +.019 Capill.; −.076 to 32° = 29.756.

| | ° | | m s | m s | h m s | ° | ° | | | ° | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .... | .... | 49 59 | 50 18 | | | | | | | |
| 2 | 57.6 | .... | 57 02 | 57 19 | 1 57 10.33 | 0.62 | | | | | |
| 3 | .... | .... | 04 02 | 04 22 | | | 57.6 | 422.52 | .... | +0.28 | 86056.88 |
| 25 | .... | .... | 38 49 | 39 28 | | | | | | | |
| 26 | 57.6 | .... | 45 51 | 46 30 | 4 46 10.83 | 0.18 | | | | | |
| 27 | .... | .... | 52 53 | 53 54 | | | | | | | |

**Exp. C.** In rarified air. Aug. 21 P.M. to Aug. 22 A.M. Dent making 86465.92 vibrations.

| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .... | .... | 25 53 | 26 03 | | | | | | | |
| 2 | 57.4 | 1.00 | 33 05 | 33 16 | 5 36 47.25 | 1.15 | | | | | |
| 3 | .... | .... | 40 18 | 40 30 | | | 56.2 | | | | |
| 4 | .... | .... | 47 30 | 47 43 | | | + 0.75 | 434.17 | 1.95 | +0.68 | 86068.26 |
| 122 | .... | .... | 01 11 | 01 46 | | | 56.95 | | | | |
| 123 | .... | .... | 08 25 | 09 05 | | | | | | | |
| 124 | 55.0 | 2.90 | 15 40 | 16 20 | 20 12 22 | 0.16 | | | | | |
| 125 | .... | .... | 22 53 | 23 36 | | | | | | | |

**Exp. D.** In full atmospheric pressure. Aug. 22 A.M. Dent making 86465.92 vibrations.

Barom. { beginning 29.816 / ending .. 29.760 } 29.788 at 55°5 ; +.019 Capill. ; −.070 to 32° =29.737.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidences. | Arcs. | Mean Temp. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.5 | .... | 03 41 | 03 53 | 9 03 47 | 1.00 | 56.7 | 422.71 | .... | +0.67 | 86057.48 |
| 26 | .... | .... | 59 43 | 00 07 | | | | | | | |
| 27 | 56.9 | .... | 06 44 | 07 10 | 12 06 57.33 | 0.23 | | | | | |
| 28 | .... | .... | 13 46 | 14 14 | | | | | | | |

Pendulum removed. Planes examined. Pendulum replaced, Weight above.

**Exp. E.** In full atmospheric pressure. Aug. 22 P.M. Dent making 86465.92 vibrations.

Barom. { beginning 29.760 / ending .. 29.650 } 29.705 at 57°.5 ; +.019 Capill. ; .074 to 32° = 29.650.

| | ° | | m s | m s | h m s | ° | ° | | | ° | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .... | .... | 33 50 | 34 00 | | | | | | | |
| 2 | 57.5 | .... | 40 49 | 41 00 | 12 40 54.5 | 1.03 | | | | | |
| 3 | .... | .... | 47 48 | 48 00 | | | | | | | |
| 26 | .... | .... | 29 00 | 30 13 | | | 57.5 | 421.78 | .... | +0.99 | 86056.88 |
| 27 | .... | .... | 35 56 | 37 12 | | | | | | | |
| 28 | 57.5 | .... | 43 04 | 44 20 | 3 43 40.7 | 0.095 | | | | | |
| 29 | .... | .... | 50 02 | 51 26 | | | | | | | |
| 30 | .... | .... | 57 04 | 58 30 | | | | | | | |

**Exp. F.** In rarified air. Aug. 23 A.M. to P.M. Dent making 86465.92 vibrations.

| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 59.05 | | | | |
| 1 | 58.3 | 0.94 | 18 40 | 18 50 | 9 18 45 | 1.24 | + 0.75 | 433.85 | 1.045 | +1.27 | 86068.56 |
| 66 | 59.8 | 1.15 | 08 30 | 09 01 | 5 08 45.5 | 0.30 | 59.80 | | | | |

**Exp. G.** In full atmospheric pressure. Aug. 24 A.M. Dent making 86466.00 vibrations.

Barom. { beginning 29.200 / ending .. 29.360 } 29.280 at 60° ; +.019 Capill. ; −.082 to 32° = 29.217.

| | ° | | m s | m s | h m s | ° | ° | | | ° | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 59.8 | .... | 4 32 | 4 42 | 8 4 37 | 1.19 | 59.6 | 420.52 | .... | +1.06 | 86055.82 |
| 26 | 59.4 | .... | 59 24 | 00 16 | 10 59 50 | 0.12 | | | | | |

The ends of the bar of the pendulum were then blackened with lamp-black and varnish, to make a stronger contrast with the white disc on the clock pen-

dulum. The slider was moved to 1.633 inch from the middle of the pendulum towards the weight. The pendulum was then replaced, and the experiments were commenced for determining the rate due to the distance between the axes of suspension.

Slider at 1.633 inch from the middle towards the weight. Weight below.

**Exp. H.** August 24 P.M. to August 25 A.M. Dent making 86466.29 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidences. | Arcs. | Mean Temp. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 58.45 | 0.965 | 42 57 | 43 07 | 5 43 02 | 1.24 | 56.62 + 0.75 | 433.70 | 1.207 | +0.95 | 86068.49 |
| 117 | 54.8 | 1.450 | 41 16 | 41 46 | 19 41 31 | 0.26 | 57.37 | | | | |

**Exp. I.** August 25 A.M. to August 25 P.M. Dent making 86466.31 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 55.0 | 0.76 | 09 53 | 10 04 | 20 09 58.5 | 1.08 | 55.86 + 0.75 | 434.15 | 0.930 | +1.05 | 86069.04 |
| | 55.4 | .... | .... | .... | 22 0 0 | .... | 56.61 | | | | |
| | 56.0 | .... | .... | .... | 24 0 0 | .... | | | | | |
| | 56.4 | .... | .... | .... | 2 0 0 | .... | | | | | |
| 78 | 56.5 | 1.10 | 26 56 | 27 20 | 5 27 08 | 0.42 | | | | | |

**Exp. K.** August 25th P.M. to August 26th A.M. Dent making 86466.33 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.6 | 1.10 | 32 23 | 32 34 | 5 32 28.5 | 1.22 | 55.50 + 0.75 | 434.10 | 1.365 | +0.85 | 86068.80 |
| 122 | 54.4 | 1.63 | 07 39 | 08 11 | 20 07 55 | 0.21 | 56.25 | | | | |

**Exp. L.** August 26th A.M. to August 26th P.M. Dent making 86466.15 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | .... | .... | 32 02 | 32 14 | | | | | | | |
| 2 | 55.0 | 0.685 | .... | .... | 20 39 22 | 0.99 | | | | | |
| 3 | .... | .... | 46 30 | 46 42 | | | 55.92 | | | | |
| | 55.8 | .... | .... | .... | 23 45 0 | | 0.75 | 434.42 | 0.938 | +0.93 | 86069.01 |
| | 56.4 | .... | .... | .... | 2 45 0 | | 56.67 | | | | |
| 74 | .... | .... | 20 29 | 20 52 | | | | | | | |
| 75 | 56.5 | 1.190 | 27 43 | 27 06 | 5 27 55 | 0.40 | | | | | |
| 76 | .... | .... | 34 58 | 35 22 | | | | | | | |

**Exp. M.** August 26th P.M. to August 27th A.M. Dent making 86465.98 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Resp. | Times of Coincidences. | Arcs. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.7 | 1.24 | 51 10 | 51 19 | 5 51 14.5 | 1.27 | 56.73 | 433.55 | 1.615 | +0.91 | 86067.99 |
| 118 | 56.75 | 1.99 | 56 22 | 56 57 | 19 56 39.5 | 0.21 | + 0.75 = 57.48 | | | | |

**Exp. N.** August 27th A.M. Dent making 86466.06 vibrations.

Barom. { beginning 29.520 57.5 / ending .. 29.508 58.0 } 29.514 ; +.019 Capill. ; −0.76 to 32° = 29.457.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 57.5 | .... | 47 23 | 47 31 | 20 47 27 | 1.17 | 57.50 | 421.61 | .... | +0.95 | 86056.84 |
| 24 | .... | .... | 28 51 | 29 15 | | | | | | | |
| 25 | 57.5 | .... | .... | .... | 23 36 05.75 | 0.30 | | | | | |
| 26 | .... | .... | 42 58 | 43 19 | | | | | | | |

The Pendulum was then removed; the planes examined, and found truly horizontal; and the Pendulum replaced with the Weight above.

**Exp. O.** In Atmospheric Pressure. August 28th A.M. Dent making 86466.25 vibrations.

Barom. { beginning 29.382 / ending . 29.470 } 29.426 55°.5 ; +.019 Capill. ; −.069 to 32° = 29.376.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Resp. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.0 | .... | 18 52 | 19 02 | 20 18 57 | 1.20 | 56.05 | 420.33 | .... | +1.11 | 86055.95 |
| 25 | 56.1 | .... | 06 35 | 07 35 | 23 07 05 | 0.13 | | | | | |

**Exp. P.** August 28th P.M. Dent making 86466.35 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 55.5 | 0.895 | 46 52 | 47 01 | 00 50 32.25 | 1.29 | 55.95 | 432.88 | 1.442 | +1.82 | 86068.67 |
| 2 | 55.7 | .... | 54 03 | 54 13 | | | + 0.75 = 56.70 | | | | |
| 42 | 56.3 | .... | 42 32 | 42 56 | 5 46 20.25 | 0.48 | | | | | |
| 43 | 56.3 | 1.990 | 49 45 | 50 08 | | | | | | | |

**Exp. Q.** August 28th P.M. Dent making 86466.40 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | 55.95 | | | | |
| 1 | 56.3 | 1.13 | 14 33 | 14 43 | 6 14 38 | 1.38 | + 0.75 | 432.33 | 1.650 | +2.07 | 86068.45 |
| 40 | 55.6 | 2.17 | 55 24 | 55 54 | 10 55 39 | 0.51 | 56.70 | | | | |

**Exp. R.** August 28th P.M. to August 29th A.M. Dent making 86466.44 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | 55.80 | | | | |
| 1 | 55.6 | 0.85 | 9 24 | 9 37 | 11 09 30.5 | 1.32 | + 0.75 | 433.29 | 1.685 | +1.15 | 86068.45 |
| 72 | 56.0 | 2.52 | 41 51 | 42 37 | 19 42 14 | 0.20 | 56.55 | | | | |

**Exp. S.** August 29th A.M. to August 29th P.M. Dent making 86466.43 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | 56.80 | | | | |
| 1 | 56.0 | 0.935 | 3 13 | 3 35 | 20 3 24 | 1.25 | + 0.75 | 432.89 | 1.512 | +1.40 | 86068.34 |
| 53 | 57.6 | 2.09 | 18 19 | 18 49 | 2 18 34 | 0.34 | 57.55 | | | | |

Slider removed to 1.566 &c. from the middle towards the weight. Planes examined, and found horizontal. Pendulum replaced, Weight above.

**Exp. T.** August 29th P.M. Dent making 86466.40 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | 57.25 | | | | |
| 1 | 58.0 | 0.985 | 42 13 | 42 23 | 4 42 18 | 1.20 | + 0.75 | 433.26 | 1.222 | +1.50 | 86068.76 |
| 49 | 56.5 | 1.460 | 28 37 | 29 12 | 10 28 54.5 | 0.41 | 58.00 | | | | |

**Exp. U.** August 29th P.M. to August 30th A.M. Dent making 86466.39 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | 56.25 | | | | |
| 1 | 56.6 | 0.960 | 41 17 | 41 27 | 10 41 22 | 1.32 | + 0.75 | 433.81 | 1.460 | +1.09 | 86068.83 |
| 82 | 55.9 | 1.960 | 26 35 | 27 27 | 20 27 01 | 0.17 | 57.00 | | | | |

**Exp. V.** August 30th A.M. to August 30th P.M. Dent making 86466.32 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Resp. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 55.9 | 1.38 | 34 39 | 34 48 | 20 34 43.5 | 1.38 | 55.85 | 433.57 | 1.115 | +1.78 | 86069.24 |
| 57 | 55.8 | 0.40 | 19 10 | 19 37 | 3 19 23.5 | 0.40 | + 0.75 | | | | |
| | | | | | | | 56.60 | | | | |

Pendulum removed and replaced. Weight below.

**Exp. W.** August 30th P.M. to August 31st A.M. Dent making 86466.26 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.7 | 0.76 | 52 59 | 53 06 | 4 53 02.5 | 1.43 | 56.10 | 433.63 | 0.95 | +1.26 | 86068.72 |
| 126 | 55.5 | 1.14 | 56 12 | 56 41 | 19 56 26.5 | 0.30 | + 0.75 | | | | |
| | | | | | | | 56.85 | | | | |

**Exp. X.** August 31st A.M. to August 31st P.M. Dent making 86466.26 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 55.5 | 0.78 | 05 00 | 05 10 | 20 05 05 | 1.22 | 55.65 | 433.92 | 1.055 | +1.02 | 86068.78 |
| 109 | 55.8 | 1.33 | 05 49 | 06 27 | 9 06 08 | 0.30 | + 0.75 | | | | |
| | | | | | | | 56.40 | | | | |

**Exp. Y.** August 31st P.M. to September 1st A.M. Dent making 86466.31 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | .... | .... | 23 33 | 23 45 | | | | | | | |
| 2 | 56.0 | 0.80 | 30 44 | 30 56 | 9 30 50.33 | 1.25 | 55.75 | 433.94 | 1.165 | +1.03 | 86068.82 |
| 3 | .... | .... | 37 56 | 38 08 | | | + 0.75 | | | | |
| 109 | .... | .... | 24 27 | 24 58 | | | | | | | |
| 110 | 55.5 | 1.53 | 31 42 | 32 10 | 22 31 56.17 | 0.29 | 56.50 | | | | |
| 111 | .... | .... | 38 55 | 39 25 | | | | | | | |

**Exp. Z.** September 1st A.M. to September 1st P.M. Dent making 86466.29 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 55.6 | 0.82 | 48 11 | 48 20 | 22 51 51 | 1.26 | 56.2 | 433.55 | 0.985 | +1.28 | 86068.70 |
| 2 | .... | .... | 55 22 | 55 32 | | | + 0.75 | | | | |
| 87 | .... | .... | 9 26 | 9 52 | 9 13 16.75 | 0.41 | | | | | |
| 88 | 56.8 | 1.15 | 16 41 | 17 08 | | | 56.95 | | | | |

**Exp. A A.** September 1st P.M. to September 2nd P.M. Dent making 86466.24 vibrations.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Resp. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 57.0 | 0.665 | 23 35 | 23 44 | 9 23 39.5 | 1.40 | 56.55 | | | | |
| 132 | 56.1 | 1.140 | 9 58 | 10 25 | 25 10 11.5 | 0.30 | + 0.75 | 433.53 | 0.90 | +1.23 | 86068.56 |
| | | | | | | | 57.30 | | | | |

**Exp. B B.** September 2nd P.M. Dent making 86466.24 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 57.0 | 0.72 | 15 52 | 16 01 | 1 15 56.5 | 1.30 | 56.85 | | | | |
| 76 | 56.7 | 1.07 | 16 54 | 17 18 | 10 17 06 | 0.55 | + 0.75 | 432.93 | 0.895 | +1.63 | 86068.41 |
| | | | | | | | 57.60 | | | | |

**Exp. C C.** September 2nd P.M. to September 3rd A.M. Dent making 86466.24 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.7 | 0.70 | 25 49 | 26 02 | 10 25 55.5 | 1.20 | 56.05 | | | | |
| 80 | 55.4 | 1.14 | 56 33 | 56 55 | 19 56 44 | 0.48 | + 0.75 | 433.53 | 0.92 | +1.33 | 86068.67 |
| | | | | | | | 56.80 | | | | |

**Exp. D D.** September 3rd A.M. to September 3rd P.M. Dent making 86466.23 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 55.5 | 0.80 | 06 37 | 06 46 | 20 06 41.5 | 1.30 | 55.75 | | | | |
| 79 | 56.0 | 1.34 | 30 17 | 30 44 | 5 30 30.5 | 0.48 | + 0.75 | 433.70 | 1.07 | +1.47 | 86068.97 |
| | | | | | | | 56.50 | | | | |

**Exp. E E.** September 3rd P.M. to September 4th A.M. Dent making 86466.23 vibrations.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | in. | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 56.0 | 0.89 | 40 31 | 40 40 | 5 40 35.5 | 1.30 | 55.75 | | | | |
| 147 | 55.5 | 1.17 | 16 42 | 17 17 | 23 16 59.5 | 0.19 | + 0.75 | 434.14 | 1.03 | +0.99 | 86068.89 |
| | | | | | | | 56.50 | | | | |

The air was then admitted, and the following observations made to obtain the reduction to a vacuum, for the small portion of air remaining in the apparatus, indicated by the gauge, in those of the preceding experiments in which the weight was below.

Slider at 1.566. Weight below.

EXP. F F. September 7th A.M. to September 7th P.M. DENT making 86465.83 vibrations.

Barom. { beginning 29.422 57.5 / ending . 29.420 60 } 29.421 59°; + .019 Capill.; −.079 reduction to 32° = 29.361.

| No. of Coincid. | Therm. | Gauge. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arc of Vibration. | Mean Therm. | Mean Interval. | Mean Gauge. | Correction for Arc. | Vibrations in 24 hours. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 57.0 | .... | 09 05 | 09 13 | 10 09 09 | 1.22 | 57.5 | 422.00 | .... | +0.85 | 86056.88 |
| 31 | 58.0 | .... | 39 54 | 40 24 | 1 40 09 | 0.21 | | | | | |

EXP. G G. September 7th P.M. DENT making 86465.80 vibrations.

Barom. { beginning 29.420 60 / ending . 29.382 60 } 29.401 60°; + .019 Capill.; − .083 reduction to 32° = 29.337.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | | m s | m s | h m s | ° | ° | | | ° | |
| 1 | 58.0 | .... | 50 37 | 50 47 | 1 50 42 | 1.18 | 58.0 | 421.73 | .... | +0.79 | 86056.53 |
| 31 | 58.0 | .... | 21 18 | 21 50 | 5 21 34 | 0.20 | | | | | |

Mr. TAYLOR'S *memorandum of the computation of the rate of the Clock by* DENT.

The clock's rate in these experiments has been deduced from daily comparisons between it and the Greenwich transit clock; the daily rate of the latter being determined from the observations of several stars and of the sun. With reference to these observations, it may be necessary to remark that they are made by three several observers, as is sufficiently shown in the Greenwich observations by the initials of each observer's name; and in making use of these for determining right ascensions, or the rate of the clock, it becomes necessary to apply to each observed transit the constant difference which is found to exist between the observer, and the observer of the same star on the following day, with which it is compared: for this purpose I have employed the difference as pointed out by a great many observations made for the express purpose; whence it appears that if the observer T. noted the passage of a star at the time $\tau$, the observers T. T. and N. would observe it to pass at $\tau + 0^s.16$ and $\tau + 0^s.35$ respectively. Making use of these, the rate of the Greenwich transit clock comes out:

| 1829. | s | | s |
|---|---|---|---|
| August 21 to 24 | −0.36 | September 1 | −0.24 |
| ——— 25 | −0.22 | ——— 3 | −0.32 |
| ——— 26 | −0.18 | ——— 4 | −0.20 |
| ——— 27 | −0.42 | ——— 6 | −0.38 |
| ——— 28 | −0.25 | ——— 7 | −0.31 |
| ——— 29 | −0.34 | ——— 8 | −0.11 |
| ——— 30 | −0.15 | | |
| ——— 31 | −0.24 | Mean . | −0.266 |

From the near accordance of these results with one another, and from a consideration of the inability of two or three observations (which could only be taken in some cases, on account of the cloudy weather) to determine very correctly the rate, I have preferred taking the general mean 0ˢ.27 rather than each individual result. This was accordingly employed with the following comparisons.

*Comparison of the Clock* DENT *with the Greenwich Transit Clock.*

| 1829. | Dent. d | h | m | Transit Clock. h | m | s | Daily rate of Dent. m | s | Rates employed. m | s | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| August | 20 | 21 | 52 | 7 | 51 | 50.44 | +1 | 5.92 | +1 | 5.92 | Aug. 23 |
| ——— | 24 | 5 | 24 | 15 | 33 | 13.45 | 1 | 6.66 | 1 | 6.29 | — 24 |
| ——— | 24 | 22 | 9 | 8 | 20 | 11.95 | 1 | 6.01 | 1 | 6.33 | — 25 |
| ——— | 25 | 22 | 37 | 8 | 51 | 5.40 | 1 | 5.96 | 1 | 5.98 | — 26 |
| ——— | 26 | 23 | 16 | 9 | 33 | 0.20 | 1 | 6.32 | 1 | 6.14 | — 27 |
| ——— | 27 | 23 | 40 | 9 | 59 | 52.88 | 1 | 6.60 | 1 | 6.46 | — 28 |
| ——— | 28 | 23 | 11 | 9 | 33 | 39.00 | 1 | 6.18 | 1 | 6.39 | — 29 |
| ——— | 29 | 22 | 28 | 8 | 53 | 23.90 | 1 | 6.34 | 1 | 6.26 | — 30 |
| ——— | 30 | 23 | 6 | 9 | 34 | 18.20 | 1 | 6.28 | 1 | 6.31 | — 31 |
| Sept. | 1 | 0 | 3 | 10 | 34 | 14.80 | 1 | 6.19 | 1 | 6.24 | Sept. 1 |
| ——— | 2 | 0 | 8 | 10 | 42 | 5.30 | 1 | 6.28 | 1 | 6.23 | — 2 & 3 |
| ——— | 3 | 23 | 28 | 10 | 7 | 40.33 | 1 | 6.19 | 1 | 6.23 | — 4 & 5 |
| ——— | 6 | 0 | 0 | 10 | 45 | 24.00 | | | | | |
| Clock was stopped for the re-adjustment of the disc. | | | | | | | | | | | |
| ——— | 7 | 2 | 41 | 13 | 30 | 44.20 | 1 | 6.09 | 1 | 6.14 | — 7 |
| ——— | 8 | 2 | 59 | 13 | 51 | 36.40 | | | 1 | 6.14 | — 8 |

The column entitled 'Rates employed' will be found to be the mean between the two daily rates in the preceding column; these are those which the clock may be supposed to have attained at midnight, the middle time between the two comparisons; but since the comparisons were taken in the middle of

the day, with observations of coincidences before and after, it will be advisable to take the mean of these daily rates as above for the rate to be employed throughout on any one day, rather than employ the former for coincidences observed in the morning, and the latter for afternoon coincidences.

(Signed) THOMAS GLANVILLE TAYLOR.

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Experiments with Captain KATER's Pendulum on the fixed iron frame on the south wall of the Pendulum Room at the Royal Observatory. The agate planes the same which had been used in the vacuum apparatus. The same thermometer also, and arc. Barometer, the standard of the Observatory. Slider at 1.6. Weight below. Clock by GRAHAM; its rate furnished by Mr. THOMAS GLANVILLE TAYLOR in the subjoined memorandum. The correction for the arcs computed by the usual formula, and multiplied by 1.13. (p. 469.)

EXP. H H. September 30. GRAHAM making 86520.71 vibrations.

Barom. {30.110 51 / 30.116 51} 30.113; +.019 Capill.; −.058 to 32° = 30.074.

| No. of Coincid. | Therm. | Times of Disap. | Times of Resp. | Times of Coincidence. | Arcs of Vibration. | Mean Temp. | Mean Interval. | Correction for Arc. | Reduc$^n$ to mean temp. 53° | Reduction to a vacuum. | Vibrations in vacuo at 53°. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | m s | m s | h m s | ° | | | | | | |
| 1 | .... | 3 33 | 3 43 | | | | | | | | |
| 2 | 49.65 | 9 47 | 9 58 | 11 09 52.3 | 0.80×1.05=0.84 | | | | | | |
| 3 | .... | 16 00 | 16 13 | | | 50.5 | 375.38 | +0.43 | −1.10 | +12.30 | 86071.35 |
| 30 | .... | 4 49 | 5 16 | | | | | | | | |
| 31 | 51.35 | .... | .... | 2 11 18.2 | 0.19×1.05=0.20 | | | | | | |
| 32 | .... | 17 20 | 17 48 | | | | | | | | |

EXP. I I. Fresh impulse. Barom. {30.118 51.25 / 30.144 51.25} 30.131; +.019 Capill.; −.058 to 32° = 30.092.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 51.3 | 35 47 | 35 56 | 2 35 51.5 | 0.80×1.05=0.84 | 51.1 | 375.02 | +0.54 | −0.84 | +12.30 | 86071.28 |
| 24 | 50.9 | 59 24 | 59 50 | 4 59 37 | 0.28×1.05=0.29 | | | | | | |

EXP. K K. October 1. GRAHAM making 86520.71 vibrations.

Barom. {30.150 50 / 30.152 54} 30.151; +.019 Capill.; −.060 to 32° = 30.110.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 50.8 | 41 33 | 41 40 | 9 41 36.5 | 0.81×1.05=0.85 | 51.65 | 375.10 | +0.47 | −0.59 | +12.28 | 86071.52 |
| 31 | 52.5 | 48 57 | 49 22 | 0 49 09.5 | 0.21×1.05=0.22 | | | | | | |

**Exp. L L.** Fresh impulse. Barom. $\left\{\begin{matrix}30.152\\30.115\end{matrix}\right\}$ 30.133 at 55°; +.019 Capill.; −.069 to 32° = 30.083.

| No. of Coincid. | Therm. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arcs of Vibration. | Mean Temp. | Mean Interval. | Correction for Arc. | Reduc$^{n}$ to mean temp. 55° | Reduction to a Vacuum. | Vibrations in Vacuum at 55°. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 54.8 | 17 11 | 17 22 | 1 17 16.5 | 0.80×1.05 = 0.84 | 53.8 | 373.91 | +0.39 | +0.35 | +12.21 | 86070.85 |
| 36 | 52.8 | 55 15 | 55 32 | 4 55 23.5 | 0.15×1.05 = 0.16 | | | | | | |

**Exp. M M.** October 3. Graham making 86520.80 vibrations.

Barom. $\left\{\begin{matrix}29.697\\29.650\end{matrix}\right\}$ 29.673 at 57°; +.019 Capill.; −.072 to 32° = 29.620.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 55.0 | 25 36 | 25 48 | 10 25 42 | 0.78×1.05 = 0.82 | 54.8 | 373.39 | +0.36 | +0.79 | +12.02 | 86070.53 |
| 36 | 54.6 | 03 23 | 03 38 | 2 03 30.5 | 0.15×1.05 = 0.16 | | | | | | |

**Exp. N N.** Fresh impulse. Barom. $\left\{\begin{matrix}29.650\\29.648\end{matrix}\right\}$ 29.649 at 56°; +.019 Capill.; −.070 to 32° = 29.598.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 54.7 | 8 51 | 9 01 | 2 8 56 | 0.84×1.05 = 0.88 | 54.55 | 373.5 | +0.50 | +0.68 | +12.01 | 86070.71 |
| 30 | 54.4 | 9 15 | 9 40 | 5 9 27.5 | 0.22×1.05 = 0.23 | | | | | | |

**Exp. O O.** October 4. Graham making 86520.89 vibrations.

Barom. $\left\{\begin{matrix}29.860\\29.856\end{matrix}\right\}$ 29.858 at 52°; +.019 Capill.; −.061 to 32° = 29.816.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 52.5 | 01 25 | 01 34 | 10 01 29.5 | 0.82×1.05 = 0.86 | 53.1 | 374.16 | +0.39 | +0.04 | +12.12 | 86070.97 |
| 40 | 53.7 | 04 31 | 04 52 | 2 04 41.5 | 0.14×1.05 = 0.15 | | | | | | |

**Exp. P P.** Fresh impulse. Barom. $\left\{\begin{matrix}29.856\\29.850\end{matrix}\right\}$ 29.853 at 54°; +.019 Capill.; −.066 to 32° = 29.806.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 53.7 | 10 37 | 10 47 | 2 10 42 | 0.77×1.05 = 0.81 | 54.0 | 373.80 | +0.49 | +0.44 | +12.11 | 86071.03 |
| 28 | 54.3 | 58 41 | 59 08 | 4 58 54.5 | 0.23×1.05 = 0.24 | | | | | | |

**Exp. Q Q.** October 5. Graham making 86520.93 vibrations.

Barom. $\left\{\begin{matrix}29.470\\29.472\end{matrix}\right\}$ 29.471 at 55°; +.019 Capill.; −.068 to 32° = 29.422.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m s | m s | h m s | | | | | | | |
| 1 | 54.2 | 40 43 | 40 55 | 10 40 49 | 0.80×1.05 = 0.84 | 54.25 | 373.73 | +0.36 | +0.55 | +11.93 | 86070.74 |
| 40 | 54.3 | 43 35 | 43 54 | 2 43 44.5 | 0.13×1.05 = 0.14 | | | | | | |

EXP. R R. Fresh impulse. Barom. $\left\{\begin{matrix}29.472\\29.500\end{matrix}\right\}$ 29.486 at 53°.5 ; +.019 Capill. ; −.065 to 32° = 29.440.

| No. of Coincid. | Therm. | Times of Disap. | Times of Reap. | Times of Coincidence. | Arcs of Vibration. | Mean Temp. | Mean Interval. | Correction for Arc. | Reduc$^{n}$ to mean temp. 55° | Reduction to a Vacuum. | Vibrations in Vacuum at 58°. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ° | m s | m s | h m s | ° | ° | s | s | s | s | |
| 1 | 54.4 | 50 07 | 50 17 | 2 50 12 | 0.80×1.05=0.84 | 53.7 | 373.80 | +0.35 | +0.30 | +11.97 | 86070.64 |
| 43 | 53.0 | 11 39 | 12 04 | 7 11 51.5 | 0.12×1.05=0.13 | | | | | | |

Mr. TAYLOR'S *memorandum of the computation of the rate of the Clock by* GRAHAM.

The rate of the clock by GRAHAM has been determined by comparisons between it and the Greenwich transit clock, made at intervals of twelve hours nearly; the rate of the latter being determined by transits of the sun and of several stars, from which it appears that the transit clock's rate was,

| | | | | |
|---|---|---|---|---|
| 1829. September | 29 | evening to September | 30 evening | +0.06 |
| ——— | 30 | ——— to October | 1 ——— | −0.14 |
| October | 1 | ——— to ——— | 2 ——— | −0.04 |

These being employed with the following comparisons give the rate of the clock GRAHAM:

| | Graham. h m | Transit Clock. h m s | Rate of Graham. m s |
|---|---|---|---|
| September 30 morning | 18 33 | 11 35 27.05 | +2 0.76 |
| ——— 30 evening | 6 23 | 23 26 14.10 | 2 0.63 |
| October 1 morning | 19 20 | 12 24 26.48 | 2 0.79 |
| ——— 1 evening | 6 33 | 23 38 20.45 | 2 0.67 |
| ——— 2 morning | 18 58 | 12 4 20.30 | |

The very near accordance of these results with their mean $+2^{m}\ 0^{s}.71$, will seem to justify the use of this quantity for the experiments of September 30 and October 1.

Daily rate of clock GRAHAM, October 3 to 6.

| | | | |
|---|---|---|---|
| | | | m s |
| 1829. October | 3 | noon to 4 noon | +2 0.80 |
| ——— | 4 | ——— to 5 ——— | 2 0.89 |
| ——— | 5 | ——— to 6 ——— | 0.93 |

(Signed) THOMAS GLANVILLE TAYLOR.

## XXVI. *On the Sources and Nature of the Powers on which the Circulation of the Blood depends.* *By* A. P. W. Philip, *M.D. F.R.S. L. & E.*

Read June 16, 1831.

It is remarkable that, notwithstanding the great importance of the circulation in the animal economy, the length of time which has elapsed since its discovery, and the constant attention it has obtained, there is hardly any department of physiology respecting which there appears to be greater uncertainty and contrariety of opinion than the sources and the nature of the powers on which this function depends. I propose in the following paper, by comparing the principal facts on the subject, and by such additional experiments as seem still to be required, to endeavour to determine these points. Much has lately been written and many experiments have been made with this view, and it has become customary to look for the causes which support the circulation to other sources beside the powers of the heart and blood-vessels.

It has been supposed that what has been called the resilience of the lungs, that is, their tendency to collapse, by relieving the external surface of the heart from some part of the pressure of the atmosphere, is a principal means of causing it to be distended with blood, the whole weight of the atmosphere acting on its internal surface through the medium of the blood which is thus propelled from the veins into its cavities; and in this way it has been supposed that the motion of the blood through the whole of the venous part of the circulation is maintained. A similar effect has been ascribed to the act of inspiration, which it is evident must operate on the same principle; and this opinion has even been sanctioned by the Report of a Committee of the Royal Academy of Sciences of Paris *, and in this country by men whose authority is deservedly high; and the effect of these causes, it is asserted, is increased by the elastic power of the heart itself.

* Report on Dr. Barry's paper, by Baron Cuvier and Professor Dumeril.

However successfully such opinions might be combated by reasoning on the data we already possess, as direct experiment is the most simple as well as decisive way of determining the question, as reasoning on physiological subjects has so often deceived, and the experiments may here be made on the newly dead animal, and consequently without suffering of any kind, I have thought it better that the point should be determined in this way, especially as it is by experiments, which at first view seem to countenance the foregoing opinions, that their supporters attempt to establish them, with the effect, as it appears to me, of withdrawing the attention from the powers on which the circulation actually depends, and introducing considerable confusion respecting a question so immediately connected with the phenomena and treatment of disease.

With a view, therefore, to submit the foregoing opinions to this test, the following experiments were made, in which Mr. CUTLER was so good as to assist me.

EXP.—A rabbit was killed in the usual way by a blow on the occiput, and the chest opened on both sides so as freely to admit the air. The lungs were then inflated eight or ten times in the minute by means of a pipe introduced into the trachea; the circulation was found to be vigorous. On laying bare one of the femoral arteries, it was observed to pulsate strongly; and on wounding it, the blood, of a florid colour, indicating that it had undergone the proper change in its circulation through the lungs, gushed out with great force; and on introducing the hand into the thorax, the heart was found to be alternately distended and contracted as in the healthy circulation.

EXP.—All the vessels attached to the heart in the newly dead rabbit being divided, and the heart removed, it was allowed to empty itself. Its contractions continued to recur, and in their intervals it assumed a perfectly flat shape, proving that the elasticity of the heart in this animal is so small that it cannot even maintain the least cavity after the blood is discharged.

It appears from these experiments that the circulation was vigorous when none of the causes to which the motion of the blood in the veins have been ascribed existed. In the first experiment the chest being freely opened on both sides, so that the play of the lungs on inflating them could be seen, all effect on the heart either of the resilience of the lungs or the act of inspiration, was evidently prevented; and in the second, it was proved that no sensible

elasticity of the heart existed; yet while artificial respiration was performed we could perceive no abatement in the vigour of the circulation.

It is to be observed, that all these means can act only in one way in promoting the circulation, namely, by giving to the heart the power of suction; that is, by producing a tendency to vacuum in its cavities, in consequence of which the pressure of the atmosphere propels the blood from the veins into them, that of the arteries being prevented from returning to the heart by the valves at their origins. But all, as far as I know, who have either made experiments with a view to prove the supposed effect of these means on the circulation, or who have sanctioned the inferences from such experiments, have overlooked the circumstance that the veins being tubes of so pliable a nature that when empty they collapse by their own weight, whatever may be said of the effect of such causes in favouring a horizontal or descending motion of the blood, it is impossible that an ascending motion could be produced in them on the principle of suction. As far as the heart may possess any such power, its tendency must be to cause the vessel to collapse, not to raise the fluid it contains.

That the resilience of the lungs as far as they possess this property, and the act of inspiration, tend to dilate the heart and large vessels within the chest, is evident; but the former is very trifling, if it exist at all, except as far as it depends on the mere weight of the lungs; and the latter in common breathing is little more efficient, although the effect of respiration on the brain, when any part of the cranium is removed, sufficiently attests that it has a certain effect. When the breathing is so laborious as essentially to influence the circulation, it evidently tends to derange the regular flow of the blood towards the heart, inspiration of course acting interruptedly; whereas it is only necessary to inspect the chest of any of the more perfect animals immediately after death, and while artificial respiration is being performed, provided death has not been caused by great loss of blood, or an extreme and instantaneous impression on the nervous system, to see that the blood flows uniformly towards the heart with no interruption but that which the contraction of the heart itself occasions.

The elasticity of the heart is greater in some animals than in the rabbit; but it is in all cases very inconsiderable. The heart of the tortoise is the most elastic I have examined; yet even it may be compressed during its diastole by a force not sensibly greater than is sufficient to compress other muscles in a

state of relaxation. Besides, the auricles possess little or no elasticity; and whatever the elasticity of the ventricles may be, it can have no effect on the blood in the veins, because they receive their blood from the auricles which are contracting during the diastole of the ventricles. To these statements it may be added, that in many of the inferior animals the foregoing supposed causes of the venous part of the circulation evidently have no existence, and that, with the exception of the elasticity of the heart, they have no existence in the fetal state in any.

We have just seen from direct experiment, that the circulation of the blood goes on as usual when all these causes have wholly ceased to operate.

I shall now take a rapid view of the facts which, as far as I am capable of judging, leave no room of doubt respecting the sources of the power on which this function depends.

It is so evident to those in the least acquainted with the animal economy that the contractile power of the heart is one of the chief of these sources, that it would be superfluous to enumerate the proofs of it; yet even this position has been denied, and that by a writer of no mean abilities. The opposite error, however, is the more common; and not a few have ascribed, and even still do ascribe, the motion of the blood throughout the whole course of circulation to the contractile power of the heart alone, although it would not be difficult to prove that to drive the blood through one set of capillary vessels, and still more through two or three sets of such vessels,—for in man himself, in one important part of the circulation, it is carried through two, and in some animals through three, sets of capillaries before it returns to the heart,—I say it would not be difficult to prove that to drive it through one set of capillaries, at the rate at which the blood is known to move, would require a force capable of bursting any of the vessels. But here, as in the former instance, it is better to appeal to the evidence of direct facts than to any train of reasoning; and there is no want of such facts to determine the point before us, some of which I formerly had the honour to lay before the Society, and others are stated in my Treatise on the Vital Functions. The most decisive is, that the motion of the blood in the capillaries continues long after the heart has ceased to beat and the animal in the common acceptation of the term is dead, even in the warm-blooded animal, for an hour and a half or two hours, and it is not for some time sen-

sibly affected by the heart's ceasing to beat; nor does this arise from some imperceptible impulse still given by the heart, because when all the vessels attached to this organ are secured by a ligature and the heart cut out, the result is the same.

That the circulation in the capillary vessels is independent of the heart, may be shown by various other means. On viewing the motion of the blood in them, with the assistance of the microscope, it may generally be observed that it is moving with different degrees of velocity in the different vessels of the part we are viewing, frequently more than twice as rapidly in some than in others. Were the motion derived from a common source, this could not be the case. It is impossible, in the motion of the blood in the capillaries, in the least degree to perceive the impulse given by the beating of the heart, which causes the blood in the arteries to move more or less per saltum, the motion of the blood in the former being uniform as long as they retain their vigour, and the necessary supply of blood is afforded from the larger vessels. I have found by experiments very frequently repeated*, that the motion of the blood may be accelerated or retarded in the capillaries by stimulants or sedatives, acting not through the medium of the heart, but on these vessels themselves. Nay, so little effect has the action of the heart on the motion of the blood in the capillaries, that I have found that when the power of the capillaries of a part is suddenly destroyed by the direct application of opium to them, the motion of the blood in them instantly ceases, although the vigour of the heart and that of every other part of the sanguiferous system is entire †.

If the circulation in the capillaries be thus independent of the heart, it is evident that the influence of that organ cannot extend to the veins. On comparing the whole of the foregoing circumstances, is it not a necessary inference that the motion of the blood in the veins, like that in the capillaries, depends on the power of these vessels themselves? But that we may not trust to any train of reasoning, where it is possible to have recourse to direct proof, I made the following experiment, with the assistance of Mr. Cutler.

Exp.—In the newly dead rabbit, in which the circulation was maintained by artificial respiration, the jugular vein was laid bare for about an inch and a half; a ligature was then passed behind the part of the vessel nearest to the

* My Treatise on the Vital Functions. † Ibid.

head, and the animal was so placed that the vein was brought into the perpendicular position, the head of the animal being undermost, so that it was necessary for the vein, in conveying the blood to the heart, to convey it perpendicularly against its gravity. The ligature, which was placed at what was now the lowest part of the exposed portion of the vein, was suddenly tightened, while Mr. CUTLER and myself observed the vessel. The blood in the part of the vein between the ligature and the heart was instantly and completely expelled, as the transparency of the vessel enabled us to perceive. The vessel itself wholly collapsed, proving that all its blood had entered the heart, so that to a superficial view there seemed to be no vessel in the part where a large dark-coloured vein had just before appeared. In the mean time, on the other side of the ligature, the vein had become gorged with blood.

In the foregoing experiment we see the blood rising rapidly against its gravity, where all causes external to the vessel on which the venous part of the circulation has been supposed to depend, had ceased to exist, and the vis à tergo was wholly destroyed by the ligature.

By a similar experiment, the power of the arteries in propelling the blood may also be demonstrated.

EXP.—In a newly dead rabbit, the circulation being supported by artificial breathing, the carotid artery was laid bare for about an inch and a half. The animal was so placed as to keep the vessel in the perpendicular position, the head being now uppermost. A ligature was passed behind that part of the vessel which was next the heart, and Mr. CUTLER and myself observed the vessel at the moment the ligature was tightened. The artery of course did not collapse as the vein had done in the preceding experiment; but the blood was propelled along the vessel, so that it no longer appeared distended with it. It was at once evident, from the change of appearance in the vessel, that the greater part of the blood had passed on in a direction perpendicularly opposed to its gravity. It is worthy of remark, that the blood of the artery was propelled neither so rapidly nor so completely as that of the vein, the cause of which will be evident in the observations I am about to make on the nature of the function and powers of these vessels.

When the whole of the preceding facts are considered, it will, I think, be admitted that the circulation is performed by the combined power of the heart

and blood-vessels themselves, and that no auxiliary power is necessary for its perfect performance. Here, as in other cases, the more we study the operations of nature, the more direct and simple we find them. The resilient power of the lungs and elasticity of the ventricles of the heart, as far as they exist, favour the free entrance of the blood into these cavities, an office adapted to the feebleness of such powers, which, in many animals we have seen, have no existence. Their operation is similar, but probably much inferior, to the elastic power of the arteries, by which the ingress of the blood suddenly impelled into them by the systole of the heart, is rendered more free than it would have been had these vessels tended to collapse in the intervals of its contractions. Had the blood flowed into them in a continued stream, and been carried through them by their own powers alone, their elasticity would evidently have impeded, not promoted, the circulation through them. Thus the veins, where these conditions obtain, are so pliable that they collapse by their own weight, and hence it was that in the preceding experiments the vein carried on its blood so much more rapidly and completely than the artery, which felt the want of the impulse it receives from the heart, that at once assists in propelling its blood, and through the blood stimulates the vessel itself. The action of the vein was perfect; it possessed all its usual powers, which reside in itself alone.

It only remains for us to inquire into the nature of the power by which the heart and blood-vessels maintain the circulation. Respecting the nature of the power of the heart there cannot be two opinions. It is evidently a muscular power. The structure of its parietes is similar to that of other muscles, and they obey all the usual laws of the muscular fibre.

Is the power of the vessels of the same nature? This is a question which has frequently been discussed. The chief arguments which have been adduced in favour of the affirmative are, the nature of their function; the fibrous appearance observed in some of the vessels, which is more evident in some other animals than in man; and the minuteness of most of the vessels, which, if they are muscular, accounts for the difficulty with which the muscular structure is detected in them. The chief arguments against the muscularity of the vessels have been, that they could not be made to obey an artificial stimulus in the way that the heart and other muscles are found to do, and that their chemical analysis gives no evidence of fibrin. Of the latter of these ob-

jections Dr. Young observes, that a part may be muscular although it does not contain fibrin, and refers in support of this opinion to the crystalline lens. The former of these objections no longer exists, the vessels having been found to obey both stimulants and sedatives as readily as parts more evidently muscular. It appears from many experiments related in my Treatise on the Vital Functions, that the action of the capillary vessels is as easily influenced both by stimulants and sedatives as the heart itself; and although the larger vessels are not so easily excited artificially as the heart and muscles of voluntary motion, yet several physiologists have succeeded in exciting them both by mechanical and chemical agents. But there is another argument in favour of the muscularity of the vessels, which, I think, may be regarded as no less powerful. I endeavoured, in papers which I had the honour to present to the Society, and which appeared in the Philosophical Transactions for 1815, to ascertain the relation which the heart bears to the nervous system, which is different from that of the muscles of voluntary motion. It appears from the facts there adduced, that this organ is not only independent of that system, although capable of being influenced through it either by means of stimulants or sedatives, and that even to the instantaneous destruction of its power; but that it equally obeys either set of agents, whether applied to the brain or spinal marrow; while the muscles of voluntary motion obey no stimulus acting through the nervous system, unless it be applied to their nerves themselves or to the particular parts of that system from which their nerves arise. I found from repeated experiments that the vessels bear the same relation to the nervous system as the heart does, their power being independent of this system, but equally with the heart capable of being influenced by either stimulants or sedatives applied either to the brain or spinal marrow, and that even to the instantaneous destruction of their power. They in all respects bear the same relation to the nervous system with the heart, which affords the strongest argument for believing that their power is of the same nature *.

From the various facts stated or referred to in the foregoing paper, the following inferences appear to be unavoidable;—That the circulation is maintained by the combined power of the heart and blood-vessels; and that the power of both is a muscular power.

* My Treatise on the Vital Functions.

Fig. 1.

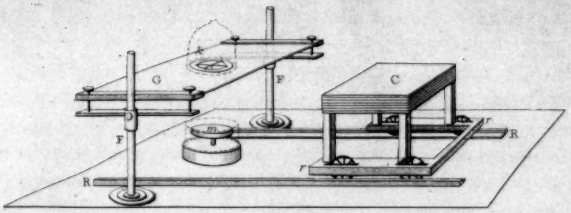
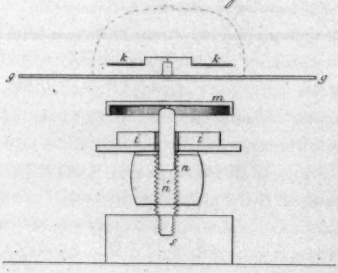



Fig. 2.



## XXVII. *On the Influence of Screens in arresting the Progress of Magnetic Action.* By WILLIAM SNOW HARRIS, *Esq. F.R.S.*

Read, June 16, 1831.

1. WHILST engaged in some experiments with a thin plate of iron, employed as a screen to intercept the action of a magnet, the following curious fact presented itself to my notice. Although the single plate of iron, which was about the $\frac{1}{10}$th of an inch in thickness, very effectually screened the action of a revolving magnet on a disc of copper, the magnet and disc being placed at a certain distance from each other; yet the same effect did not follow when the disc acted on by the magnet was also of iron: in the latter case it was found requisite to increase the quantity of intervening iron very considerably. This was done by piling several plates similar to the first, one on the other, by which the quantity of iron requisite to intercept completely the action of the magnet could be in some measure estimated.

From this I have been led to some new inquiries concerning the screening influence of substances generally, which, I trust, may possess some claim to the consideration of persons engaged in scientific pursuits. I have therefore to beg the favour of being allowed to lay them before the Royal Society.

2. Having carefully repeated the experiments from which the result just mentioned was obtained, I subsequently endeavoured to determine if the screening influence of the iron plates depended actually on the mass, or was otherwise confined to the plates immediately on the surface. With this view the interior ones were removed, and a ring of wood equal in thickness substituted in their place; the position of the upper and under plate therefore remained unchanged.

In this instance, however, the magnetic action was no longer effectually intercepted, nor could it be completely shut out, except by restoring all the interior plates as before; that is to say, two plates above and two below were insufficient, and so on.

3. From this circumstance, it seemed not unreasonable to infer, as was subsequently proved by the fact, that a screening power might possibly be obtained in a similar way, by means of substances not containing iron, provided such substances were employed in large masses, and were in any degree susceptible of a transient magnetic state; notwithstanding that from the few experiments hitherto tried, it was rather to be inferred that such substances were deficient in this peculiar property*.

4. After a few unsuccessful attempts, I succeeded in making the screening power of several substances, not supposed to contain iron, very evident; the mechanical arrangements resorted to for the purpose were similar to those already described in my paper on the transient magnetic state, of which substances are susceptible, and which has been honoured by a place in the Royal Society's Transactions; it will be unnecessary therefore to describe them again; they will however be easily understood in the detail of the following experiments.

(*a*). A circular magnetic disc *m*, Plate XIII., fig. 1, being delicately balanced on a fine central point, by means of a rim of lead, was put into a state of rotation, on a small agate cup, at the rate of 600 revolutions in a minute; and a light ring of tinned iron *k* also finely balanced on a central pivot, placed immediately over it, at about four inches distance, by means of a thin plate of glass G.

The glass plate was supported on two sliding bars, and stands F F, by which it could be set to any required height; it was also furnished with four levelling screws, passing through two clamps of wood in which the extremities of the plate were fixed, so as to be further adjusted with the required precision: when the iron began to move slowly on its pivot, by the influence of the magnet revolving below, a large mass of copper C, about three inches thick, and consisting of plates a foot square, was carefully interposed; it being sustained on a convenient carriage *r r*, moveable on a rail-way R R, in order to be easily transferred without deranging the subject of experiment.

5. The copper thus interposed soon diminished sensibly the motion of the disc *k*, and at length arrested it altogether: on again withdrawing the copper,

* See Transactions of the Royal Society for the Year 1825. p. 469.

the motion of the disc was restored, and this effect could be obtained as often as required.

In this experiment both the magnet and disc were very completely inclosed by glass shades adapted to the nature of the experiment, and were also supported on a firmly fixed base.

(*b*). When a mass of silver or zinc of about the same dimensions was substituted for the copper, a similar result ensued; the motion of the disc was completely arrested by the screening power of the intervening mass.

(*c*). The screening property evinced by these substances depended, as in the foregoing experiment with the iron plates (2), on the whole mass interposed, as subsequently appeared by removing the interior laminæ, in which case the motion of the disc was no longer impeded.

6. It may therefore be reasonably inferred that this power of intercepting magnetic action is more or less common to every class of substance, and that to render it sensible it is only requisite to employ different bodies in masses bearing some direct ratio to their respective magnetic energies.

(*d*). Thus in substituting a similar mass of lead in the above-mentioned experiments *a b*, the motion of the disc *k* could not be completely checked, and it was subsequently found requisite to increase the quantity of the intervening mass very considerably before the screening effect became sensible to any great extent; the magnetic energy of lead being so much less than that of copper.

7. The screening influence of substances is best shown by employing a powerful magnet, and by placing the disc *k* just within the limit of the action; thus a sufficient mass may be interposed, and the screening effect made very evident. To exemplify the influence of distilled water in this way at about 32° of FAHRENHEIT'S scale, I am led to believe it would be requisite to obtain a slight action on the disc, at a distance of rather more than thirty feet, so as to interpose, about the same thickness of ice.

8. This curious property seems to be intimately connected with a principle, which may be termed a neutralization of force; by which the magnetic action is, as it were controulled, as in the following experiment.

(*e*). A circular magnetic disc *m*, Fig. 2, being put into a state of rapid rotation, a light ring of copper *k* movable on a fine centre was placed imme-

diately over it as in Fig. 1, on a thin plate of glass *g g*, and at such a distance as to be just within the limit of the action; when the copper disc began to move slowly on its pivot, a mass of iron *i* about half an inch thick was carefully raised immediately under the revolving magnet, by means of a nut *n* and screw *n'* affixed to the block *s* on which the whole was sustained; when the iron *i* was near the under surface of the magnet, the action on the disc above began to diminish, so that the motion of the disc was finally checked altogether: by depressing the mass of iron *i* the motion of the disc was again restored, and this result could be obtained as often as required.

9. It does not therefore appear essential to place the iron *i* immediately between the magnet and disc in order to screen the action of the former; since the same effect is produced when the iron is placed immediately beneath the magnet.

10. This subject of screens seems to possess great scientific interest, and if fully investigated is not unlikely to bring us further acquainted with one of those wonderful agencies, on which the phænomenon of attraction may be supposed to depend, the more complete elucidation of which is of the utmost consequence in the present state of physical science.

*Plymouth, April* 3, 1831.

Fig. 3.

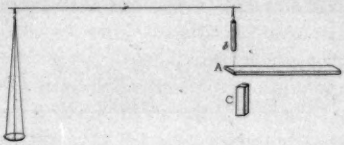



Fig. 4.

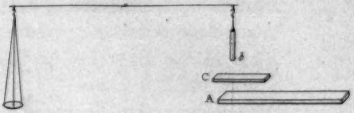
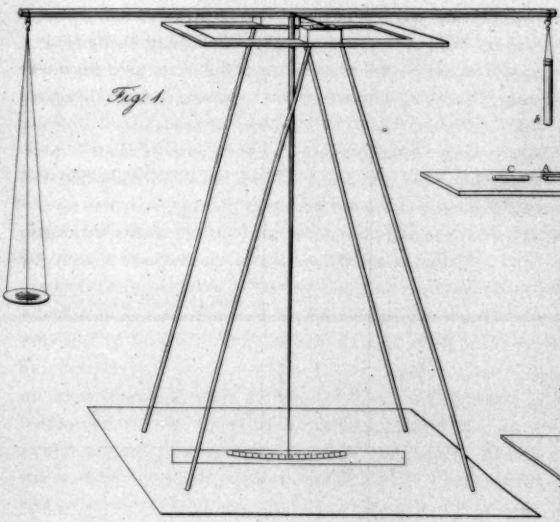



Fig. 1.



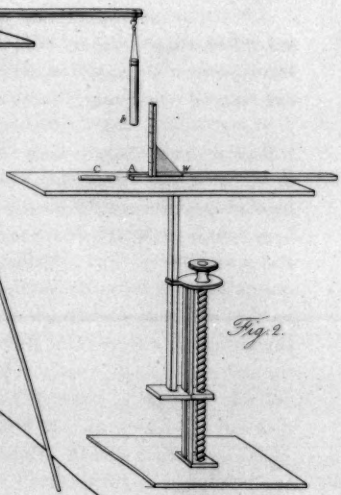
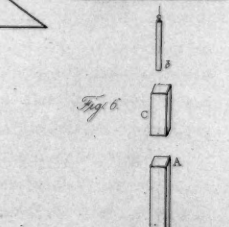



Fig. 2.



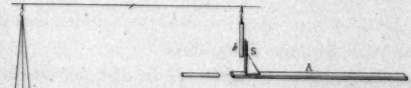
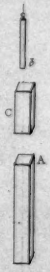



Fig. 5.

Fig. 6.



## XXVIII. *On the power of masses of Iron to controul the Attractive Force of a Magnet.* By William Snow Harris, *Esq. F.R.S.*

Read June 16, 1831.

1. It may not be unfavourable to the further elucidation of the interesting subject of screens, treated of in my last paper, and which I had the honour of laying before the Royal Society, to give a short account of some subsequent experiments concerning the effects of masses of iron on a magnet, as they have more particularly arisen out of the investigation above alluded to.

2. The principal part of the apparatus which I employed in these experiments is represented in Plate XIV., fig. 1 & 2; it is extremely simple, and will be readily understood by the following description of it.

A light beam of dry clean-grained deal, of about fourteen inches in length, and somewhat less than a quarter of an inch square, is allowed to rest freely, by means of a delicate axis, on two horizontal bars of glass; the glass bars are secured on a convenient frame and stand, and the axis of the beam, as in many similar cases, is formed of a fine sewing-needle; it was passed in this case through the opposite angles, directly at the centre. From the extremity of one of the arms is suspended a small cylindrical piece of iron, about an inch and a half long and one fifth of an inch diameter, which is counterpoised by an equivalent weight, placed in a small pan suspended from the other; the method of suspension is by light hooks and rings, so as to obtain every possible degree of motion. The under part of the centre of the beam carries an index of about a foot in length, constructed of short pieces of straw, which, being tubular, are easily secured at their extremities one within the other; an index thus formed is very straight and true. By means of this index and a graduated arc the slightest motion of the beam is apparent.

3. The beam and index may be so managed, that for a short distance the deviations from the horizontal position, with equal and very small weights placed in the pan, will correspond to equal divisions on the arc, or very nearly so.

4. To regulate the distance between a magnet or other bodies, and the suspended iron, when placed immediately under the latter, either vertically or horizontally, I employed a simple screw and nut, attached to a brass frame similar to that shown in Fig. 2: thus any required altitude could be obtained, the distance between the magnet and the suspended iron being estimated by a graduated scale resting on the magnet.

5. Wherever magnetic attraction is exerted between two bodies, it seems to be accompanied by a sort of neutralization of the same force in respect of a third substance.

(*a*). Thus if the magnet A, Fig. 1, be attracting the suspended iron *b*, the vicinity of a mass of iron C will diminish the apparent force of A upon *b*: if, therefore, when the force of A upon *b* is exerted so as just to depress the beam, the force being measured by the inclination of the index (3), we place a mass of iron C close to A, the beam will immediately tend to recover its previous position.

(*b*). This power of the iron C to controul the attractive force of the magnet upon *b*, seems only to extend to a given point within the magnet, the distance between the magnet and iron remaining the same; for if the small iron *b* be suspended over a point at some distance from the extremity, as at *w*, then the action of C will not be felt at that point *w*, except by decreasing the distance between the iron and magnet, or otherwise by increasing the neutralizing power of the iron C.

6. That this depends on a sort of action which is not inappropriately termed a neutralization of force, in regard to the suspended iron *b*, is evident from the following experiments.

(*c*). The mass of iron C may be placed immediately below A, as in Fig. 3, the effect will be the most apparent when the thickness of A is not considerable; but if the magnet A be very thick, then the neutralizing effect of C is not so evident, on account of the intervening mass of which the magnet is composed, as also on account of the iron C being kept as it were at a greater distance from the immediate surface of attraction.

(*d*). Conversely, the mass of iron C may be placed above, immediately between *b* and A as in Fig. 4; but in this case we have to take into account the inductive effect on C, by which it becomes itself a temporary magnet, and con-

sequently takes on an attractive power; the experiment should therefore be so managed as to have the distance between C and *b* such, that whilst by the intervention of C the action of A is neutralized, its induced magnetic state does not become sensible upon *b* at that distance; this will be always very evident when C is of some considerable thickness, and the previous distance of A and *b* taken just within the limit of the attraction.

7. In this case C is said to screen or stop out the attraction of A upon *b*, and this probably explains the way in which screens operate in impeding the magnetic influence. It seems therefore not unreasonable to infer, that substances, possessing the greatest inductive energy, are at the same time the most powerful neutralizers. Hence in employing various bodies as screens, those are the most efficient which are susceptible of the greatest transient magnetic state: thus zinc is more efficient than lead; copper more efficient than zinc; silver than copper, and iron the most efficient of any.

8. As the distance within the magnet, to which the neutralizing force can extend, must necessarily depend on the magnetic energy of the substance employed, it would be difficult with non-ferruginous bodies to controul any very sensible portion of the action of a magnet by placing them at its extremity, Fig. 1, or beneath it, Fig. 3, except in the latter case we suppose the magnet to be extremely thin; but by intervening a considerable mass, Fig. 4, immediately between the magnet and the substance acted on, we operate directly on the contiguous attracting surface of the bar, and thus the neutralizing effect at length becomes sensible.

9. The attractive force exerted between a magnet and a mass of iron is in the direct ratio of this neutralizing power of the iron; the distance between the magnet and the iron being the same.

(*e*). Let a magnet A, Fig. 5, of about ten inches in length, and three eighths of an inch square, be placed at some convenient distance, immediately under the suspended iron *b*, and the observed force carefully counterpoised by small weights placed in the opposite pan at *p*, so as to bring the index of the beam, Fig. 1, to zero of the graduated arc; then the neutralizing power of a few small pieces of very soft iron, *w*, *x*, *y*, *z*, about the same diameter as the magnet, and varying from a quarter of an inch to two inches in length, may be easily estimated on the graduated arc, Fig. 1, by bringing each piece successively in

contact, or very nearly so, with the extremity of the magnet *a* (3). Let these small iron cylinders be now substituted in succession for the suspended iron *b*; and being first nicely counterpoised, let the attractive forces be determined at a constant distance from the magnet A by means of the graduated scale *s*; then these respective forces will be found to be very nearly in the same ratio as the previous powers of neutralization: in a great variety of cases they were found to be exactly in the same ratio.

(*f*). Where the neutralizing power is equal, there the attractive force is also equal; thus the neutralizing power, with a given magnet, not being greater in a cylindrical mass of iron of two inches in length than in one of an inch and half in length, no difference was subsequently found in the respective forces of attraction.

10. The foregoing illustrations seem to throw some light on the manner in which magnetic action may be supposed to pervade bodies.

(*g*). Having assigned any given distance, A *b*, Fig. 6, through which we know the influence of a magnet A can extend as estimated by some sensible measure *b*, then in interposing a third substance C in the space A *b*, the latter may receive a temporary magnetic state in two ways, either by the immediate action of the magnet A upon every particle of C, or otherwise by the propagation of magnetism from particle to particle, or by both: now these operations seem to be in some inverse ratio of each other; thus when the induced magnetic energy in C is considerable, the influence of the magnet A is more or less arrested by the laminæ first acted on, which operate as screens on the succeeding ones; so that the magnetic development after a certain distance proceeds entirely by propagation from one particle to another, until it is finally as it were expended; and a body *b* which was before attracted at the distance A *b* will at the same distance now remain at rest *c*. Such is the case in interposing a screen of iron between a revolving magnet and a metallic disc; but if the body C be low in the scale of magnetic energy, then the induced magnetic state is so weak that little or no screening influence is exerted between its particles, and the body *b* may be attracted as before: hence each particle of C will owe its magnetic development to the direct operation of the exciting magnet; and it is only by the successive action of a great number of particles that we at length neutralize or cut off the magnetic force by means of such a substance employed as a

screen: it therefore follows in this instance, that whilst an inconsiderable portion of the magnetic action is neutralized, a considerable mass of the screen is pervaded; at the same time a very thin stratum only of the magnet is penetrated *b* (8).

11. The diminished influence of a magnet on a metallic disc, observed to ensue on intersecting the surface of the disc by radiating grooves varying in depth *, may possibly depend on the above-mentioned circumstance (10); for in this case we actually take away a portion of the substance in which the magnetic development takes place, and thus diminish the force. I wish, however, to be understood as speaking with some degree of reserve on this point, although the conclusion is by no means unwarranted, as in the following experiment.

(*h*). The number of vibrations of a delicately suspended bar in a given arc taken in vacuo, in two similar rings of copper of equal weights and quality as nearly as may be, did not materially differ, although one of the rings was made up of separate concentric laminæ, the other being a perfectly solid mass; whereas the removal of a very thin lamina externally from the former caused a very decided change in the number of the vibrations of the bar †.

12. The preceding inquiries appear calculated to modify in some measure our views concerning the operation of a magnet, which by experiment (*e*) is rather the patient than the agent in the production of the observed effects: it cannot therefore be considered as a purely active force, much less can it be viewed as a substance, from which emanations of an unknown subtile fluid are constantly proceeding; for it may be shown, Exp. (*c*), that a magnetic lamina of steel, supposed without sensible thickness, cannot act at the same time on two masses of iron, in every respect alike, when placed between them, and at an equal distance from each; as in this case we should have an annihilation of power as regards the magnet: hence each mass of iron, if at the same time drawn by some other force, and free to move, would drop away from the magnetized steel in opposite directions. If therefore the attractive energy of a magnet be supposed to arise out of any subtle principle emanating from it,

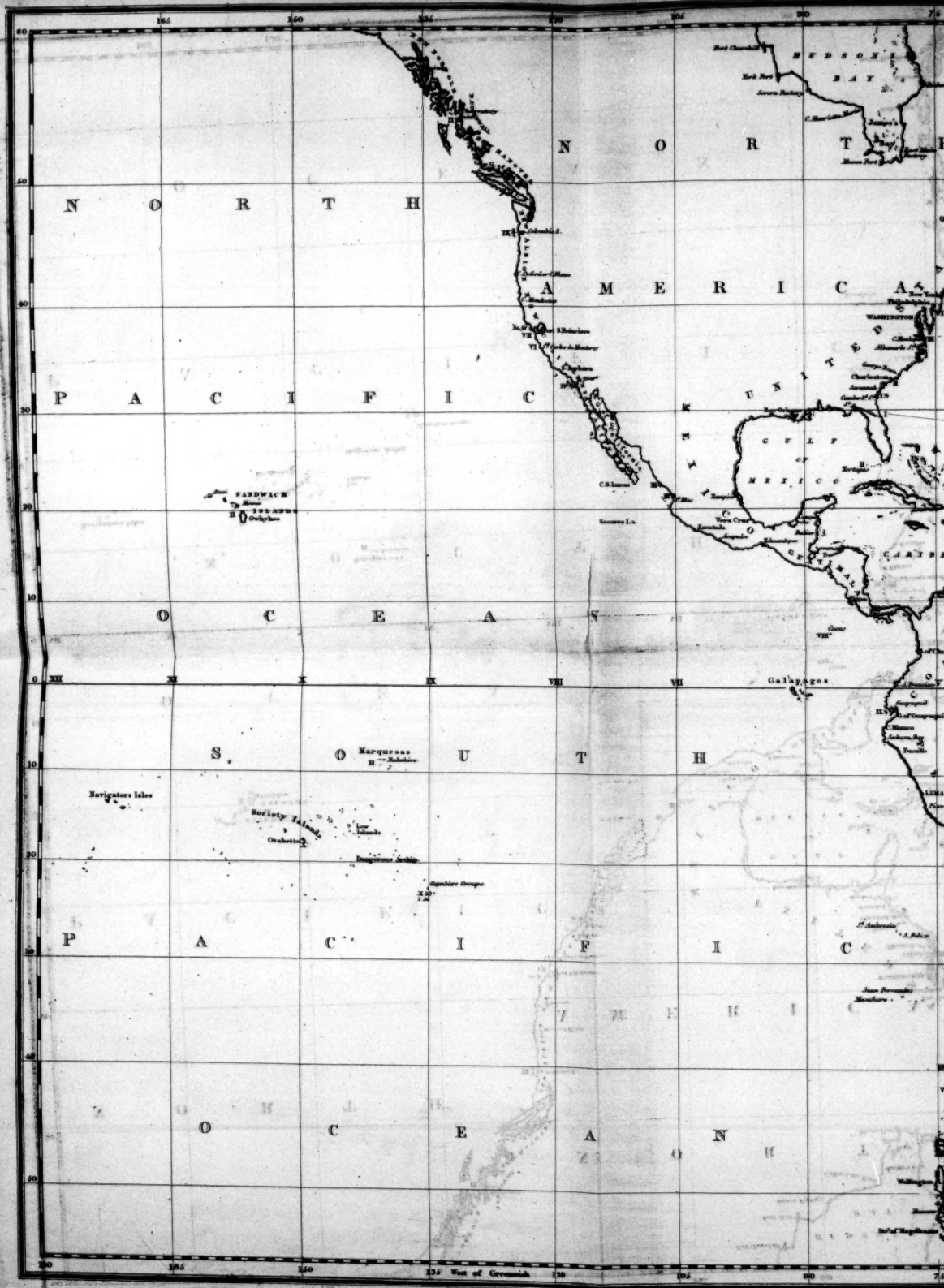
* Philosophical Transactions for 1825, p. 481.

† These vibrations were determined in the way described in the Royal Society's Transactions for 1831, Part I., p. 76.

such emanations cannot pass off in opposite directions at the same instant: now there is no sufficient reason why they should pass rather in one direction than another, and it therefore remains that an hypothesis which supposes them to pass in either, is quite unwarranted. The arrangement assumed by fine particles of iron, sifted on paper immediately over a magnet, arises out of the circumstance that the bar has generally a very sensible thickness; whilst the small particles of iron cannot operate beyond a certain distance *a*, *b*, *c*, and this equally applies to other cases in which the opposite sides of a magnet appear to attract at the same time. Moreover, the superficial boundaries of a magnet may be considered as so many distinct magnetic laminæ of uncertain thickness, as is evident from the circumstance, that the magnetic centre and poles of one surface of a bar very frequently fall in a different way from those of another surface according to the trifling variations in the progress of magnetizing; and sometimes all the surfaces differ in this respect in the same bar, that is to say, the centre and poles do not correspond to the same relative points on any two sides.

13. The wonderful phænomenon of magnetic attraction then is evidently the result of an impression first made on the magnet *e*, since with different masses of iron the attractive force at the same distance is unequal *e*: hence a magnet must be considered as a body in a peculiar state or condition, by which it may be caused to exhibit given powers or capabilities in consequence of external excitation.

15. It is always difficult in inquiries of this nature to employ terms which shall seem altogether without objection. I trust therefore that those resorted to in the course of this paper, will be taken only in the arbitrary sense in which they have been used, and not as having any necessary connection with a particular set of opinions: thus the expressions neutralizing force, magnetic development, magnetic excitation, and so on, must be taken merely as arbitrary terms, employed of necessity to facilitate the progress of inquiry, and to render its description as intelligible as possible, according to the general and unembarrassed acceptation of such terms.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Between the Latitudes of 60° N. and 55° S.
To accompany Mr Lubbock's Paper
on the
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The Roman Figures shew the Time of High Water in Greenwich Time at new and full Moon.
The Italic ——— at the Place.
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CALCUTTA
BAY OF BENGAL
AUSTRALIA
VAN DIEMEN'S LAND
NEW ZEALAND
D I A N
C E A N
East of Greenwich

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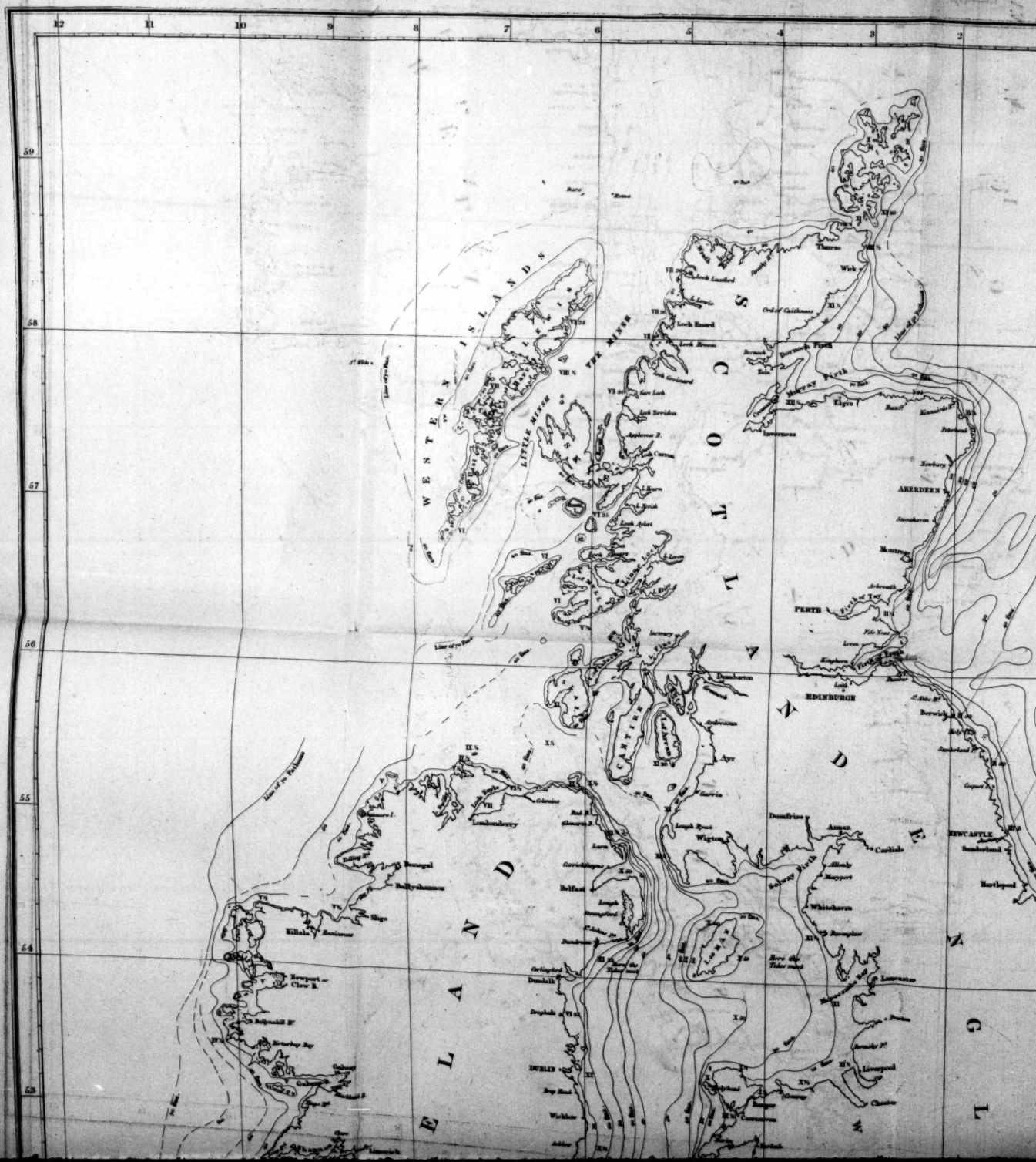
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Start Point
Plymouth
Eddystone
Falmouth
Lizard
Land's End
Scilly Islands
Bishop Rock
Longships
St Ives
Hayle
Padstow
Newquay
Tintagel
Hartland Point
Lundy Island
Ilfracombe
Barnstaple
Bideford
Clovelly
Bude
Boscastle
Tintagel
Padstow
Newquay
St Ives
Hayle
Penzance
Falmouth
Truro
Fowey
Looe
Plymouth
Dartmouth
Torbay
Exmouth
Sidmouth
Lyme
Bridport
Weymouth
Portland
Swanage
Poole
Christchurch
Lymington
Yarmouth
Isle of Wight
Cowes
Ryde
Spithead
Portsmouth
Selsea Bill
Bognor
Littlehampton
Shoreham
Brighton
Newhaven
Eastbourne
Beachy Head
Hastings
Rye
Hythe
Folkestone
Dover
Deal
Ramsgate
Margate
Sheerness
Shoeburyness
Maldon
Harwich
Orford
Southwold
Lowestoft
Yarmouth
Winterton
Cromer
Wells
Lynn Regis
Boston
Grimsby
Hull
Barmby
Whitby
Scarborough
Flamborough H.
Bridlington
Spurn Head
Holy Island
Farne Islands
Bamborough
Berwick
Sunderland
Tynemouth

# METEOROLOGICAL JOURNAL,

KEPT BY THE ASSISTANT SECRETARY

AT THE APARTMENTS OF THE

# ROYAL SOCIETY,

BY ORDER OF

## THE PRESIDENT AND COUNCIL.

MDCCCXXXI

# METEOROLOGICAL JOURNAL FOR JANUARY, 1831.

| 1831. Jan. | 9 o'clock, A.M. Barom. | 9 o'clock, A.M. Attach. Therm. | 3 o'clock, P.M. Barom. | 3 o'clock, P.M. Attach. Therm. | Dew Point at 9 A.M. in degrees of Fahr. | External Thermometer. Fahrenheit. 9 A.M. | Fahrenheit. 3 P.M. | Self-registering. Lowest. | Self-registering. Highest. | Rain, in inches. Read off at 9A.M. | Direction of the Wind at 9 A.M. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ♄ 1 | 29.766 | 38.7 | 29.782 | 41.2 | 34 | 34.6 | 42.3 | 31.5 | 42.3 | | WSW | A.M. Hoar frost—hazy. P.M. Fine and clear. |
| ☉ 2 | 29.856 | 40.3 | 29.843 | 41.6 | 36 | 38.2 | 39.3 | 33.1 | 39.7 | | SSE | Fair—lightly cloudy. |
| ☽ 3 | 29.904 | 39.7 | 29.896 | 40.9 | 38 | 38.4 | 39.8 | 32.8 | 39.8 | | S | Overcast. |
| ♂ 4 | 29.908 | 40.3 | 29.867 | 40.2 | 37 | 37.3 | 37.5 | 36.4 | 37.5 | | ESE | Foggy—light wind. |
| ☿ 5 | 29.830 | 40.3 | 29.875 | 40.8 | 37 | 37.7 | 39.2 | 36.3 | 39.2 | | E | Foggy. |
| ♃ 6 | 30.197 | 38.8 | 30.301 | 40.0 | 35 | 35.3 | 38.5 | 33.3 | 38.5 | | N | Fine and cloudless—hazy. |
| ♀ 7 | 30.592 | 36.5 | 30.580 | 37.0 | 29 | 31.0 | 32.4 | 29.3 | 33.7 | | N | Foggy—light wind. |
| ♄ 8 | 30.584 | 33.6 | 30.499 | 35.2 | 27 | 27.5 | 33.8 | 24.3 | 33.8 | | N | P.M. Hoar frost—hazy. P.M. Fine —light clouds. |
| ☉ 9 | 30.254 | 34.3 | 30.114 | 36.3 | 31 | 31.7 | 38.9 | 26.0 | 41.2 | | WNW | Foggy. |
| ☽ 10 | 29.897 | 38.9 | 29.892 | 39.8 | 40 | 40.9 | 40.8 | 30.4 | 40.8 | | NNE | Overcast—light wind and fog. |
| ♂ 11 | 30.082 | 36.7 | 30.051 | 37.8 | 33 | 33.7 | 35.2 | 32.6 | 36.3 | | ENE | Lightly cloudy and foggy. |
| ☿ 12 | 30.080 | 38.2 | 30.041 | 39.0 | 37 | 37.7 | 38.7 | 32.7 | 38.7 | | NNE | Foggy—light wind. |
| ♃ 13 | 30.120 | 38.3 | 30.128 | 39.3 | 33 | 34.7 | 38.2 | 33.2 | 38.3 | | NE | Foggy—light wind. |
| ● ♀ 14 | 30.172 | 41.3 | 30.130 | 41.9 | 37 | 39.4 | 38.3 | 33.7 | 38.5 | | NE | Foggy—light wind. |
| ♄ ♓ 15 | 29.989 | 39.3 | 29.893 | 38.2 | 32 | 32.6 | 32.4 | 31.4 | 32.6 | | E | Foggy—light wind. |
| ☉ 16 | 29.817 | 37.7 | 29.771 | 37.7 | 32 | 32.5 | 33.7 | 31.3 | 39.6 | | E | Overcast—light fog. |
| ☽ 17 | 29.605 | 39.4 | 29.565 | 40.6 | 40 | 40.7 | 43.2 | 31.6 | 43.2 | | E | Foggy—light rain and wind. |
| ♂ 18 | 29.552 | 40.3 | 29.554 | 41.9 | 38 | 38.7 | 43.2 | 36.6 | 44.3 | | E | Foggy—light rain. |
| ☿ 19 | 29.652 | 43.6 | 29.625 | 45.0 | 45 | 45.3 | 46.6 | 37.7 | 46.6 | | S | Foggy—light wind. |
| ♃ 20 | 29.411 | 44.6 | 29.273 | 45.5 | 42 | 42.3 | 43.6 | 39.7 | 43.6 | | ESE | Foggy—light rain and wind. |
| ♀ 21 | 29.131 | 45.5 | 29.091 | 47.0 | 42 | 42.6 | 47.8 | 40.3 | 47.8 | | E | Foggy—light rain and wind. |
| ♄ 22 | 29.218 | 48.3 | 29.204 | 49.2 | 47 | 47.7 | 48.5 | 41.7 | 48.5 | | E | Foggy—light rain. |
| ☉ 23 | 29.325 | 49.0 | 29.385 | 49.2 | 47 | 47.3 | 44.5 | 45.7 | 46.7 | 0.250 | E | Light rain and fog. |
| ☽ 24 | 29.488 | 42.7 | 29.502 | 41.2 | 32 | 35.7 | 35.5 | 33.1 | 35.5 | 0.083 | NNE | A.M. Snow. P.M. Light clouds and wind. |
| ♂ 25 | 29.802 | 38.8 | 29.881 | 39.3 | 30 | 33.5 | 34.2 | 30.4 | 34.2 | | NNW | Light clouds and wind. |
| ☿ 26 | 30.085 | 34.2 | 30.132 | 35.8 | 22 | 28.7 | 32.6 | 24.7 | 32.6 | | N | Fine and cloudless. |
| ♃ 27 | 29.939 | 33.6 | 29.522 | 36.0 | 21 | 29.7 | 38.8 | 25.3 | 38.8 | | SW | Overcast—light rain and wind. |
| ○ ♀ 28 | 29.650 | 37.6 | 29.656 | 38.4 | 27 | 33.0 | 35.0 | 28.7 | 35.0 | 0.194 | NNW | Fine and cloudless—light haze and wind. |
| ♄ 29 | 29.874 | 36.2 | 29.857 | 37.0 | 26 | 32.5 | 33.5 | 30.7 | 33.5 | 0.014 | N | Overcast—light wind. |
| ☉ 30 | 29.833 | 34.6 | 29.834 | 36.1 | 20 | 30.0 | 32.3 | 26.3 | 32.3 | | E | Lightly cloudy. |
| ☽ 31 | 29.727 | 33.7 | 29.623 | 36.2 | 27 | 30.5 | 33.8 | 25.2 | 33.8 | | SSE | Lightly overcast. Fall of snow early A.M. and at 1 P.M. |
| | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Sum | | |
| | 29.850 | 39.2 | 29.818 | 40.3 | 34.0 | 36.2 | 38.5 | 32.5 | 38.9 | 0.541 | | |

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.832 } { 3 P.M. 29.797 }

## OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

# METEOROLOGICAL JOURNAL FOR FEBRUARY, 1831.

| 1831. Feb. | 9 o'clock, A.M. Barom. | 9 o'clock, A.M. Attach. Therm. | 3 o'clock, P.M. Barom. | 3 o'clock, P.M. Attach. Therm. | Dew Point at 9 A.M. in degrees of Fahr. | External Thermometer. Fahrenheit. 9 A.M. | Fahrenheit. 3 P.M. | Self-registering. Lowest. | Self-registering. Highest. | Rain, in inches. Read off at 9 A.M. | Direction of the Wind at 9 A.M. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ♂ 1 | 29.099 | 33.4 | 29.047 | 36.5 | 31 | 31.9 | 34.9 | 26.7 | 35.7 | | ESE | A.M. Snow. P.M. Fine—light clouds. |
| ☿ 2 | 29.173 | 35.3 | 29.126 | 37.8 | 32 | 34.4 | 36.0 | 29.7 | 38.7 | 0.175 | SSE | Overcast. Light rain and snow P.M. |
| ♃ 3 | 29.418 | 34.9 | 29.235 | 37.0 | 33 | 34.3 | 35.9 | 28.3 | 41.4 | | S | Overcast. Light rain P.M. |
| ♀ 4 | 28.954 | 41.3 | 29.204 | 41.2 | 42 | 42.3 | 38.3 | 33.3 | 42.3 | 0.250 | E | Overcast. Foggy A.M. |
| ♄ 5 | 29.642 | 40.4 | 29.782 | 41.6 | 33 | 37.5 | 38.7 | 33.7 | 38.7 | 0.072 | W | A.M. Overcast. P.M. Fine—light clouds. |
| ☉ 6 | 29.888 | 37.6 | 29.775 | 40.3 | 30 | 31.8 | 37.7 | 29.4 | 44.4 | | SSE | Lightly overcast. |
| ☽ 7 | 29.512 | 40.7 | 29.612 | 44.4 | 46 | 46.4 | 51.2 | 30.5 | 51.7 | 0.325 | SSW | Cloudy—light wind. |
| ♂ 8 | 29.893 | 46.5 | 29.918 | 49.5 | 51 | 51.3 | 52.8 | 45.7 | 54.3 | | S | Overcast. A.M. Light wind and fog. P.M. Rain. |
| ☿ 9 | 30.004 | 51.3 | 30.054 | 54.4 | 53 | 54.5 | 58.7 | 50.7 | 59.5 | | SSW | Lightly cloudy—light wind. P.M. Fine and clear. |
| ♃ 10 | 30.234 | 53.4 | 30.231 | 55.2 | 47 | 52.6 | 57.8 | 49.3 | 57.7 | | S | Light clouds and wind. |
| ♀ 11 | 30.254 | 53.7 | 30.274 | 57.0 | 49 | 49.5 | 55.5 | 46.7 | 55.7 | | SSW | Fine—light clouds. |
| ● ♄ 12 | 30.292 | 53.7 | 30.289 | 55.2 | 50 | 50.3 | 52.8 | 48.4 | 52.8 | 0.025 | W | Fine—lightly cloudy and foggy. |
| ☉ 13 | 30.236 | 52.8 | 30.193 | 53.8 | 48 | 48.7 | 49.4 | 47.3 | 49.4 | 0.017 | SSE | Overcast—light fog. Rain P.M. |
| ☽ 14 | 30.237 | 51.5 | 30.229 | 53.0 | 45 | 46.5 | 48.6 | 45.3 | 48.6 | 0.011 | WSW | Lightly cloudy. |
| ♂ 15 | 30.131 | 48.7 | 29.984 | 50.6 | 41 | 41.0 | 48.9 | 40.3 | 48.9 | | ESE | A.M. Overcast—light fog. P.M. Fine and cloudless. |
| ☿ 16 | 29.893 | 49.7 | 29.956 | 52.5 | 46 | 46.7 | 52.0 | 40.6 | 52.3 | 0.028 | WSW | Fine—light clouds. At 3½ P.M. Heavy rain with hail. |
| ♃ 17 | 29.949 | 48.7 | 30.065 | 49.8 | 43 | 43.6 | 47.0 | 41.3 | 47.1 | 0.097 | NW | Lightly cloudy—light wind. |
| ♀ 18 | 30.201 | 46.7 | 30.176 | 49.9 | 40 | 40.6 | 46.4 | 37.1 | 46.6 | 0.014 | W | Lightly cloudy—light wind and fog. |
| ♄ 19 | 30.045 | 45.0 | 30.030 | 48.5 | 43 | 43.8 | 45.7 | 37.9 | 47.7 | | W | A.M. Foggy. P.M. Fine—light clouds and wind. Rain. |
| ☉ 20 | 29.955 | 45.3 | 29.958 | 46.3 | 36 | 41.0 | 41.8 | 38.7 | 41.8 | 0.069 | NNW | Lightly cloudy and foggy—light wind. |
| ☽ 21 | 30.053 | 41.4 | 30.045 | 44.5 | 32 | 36.8 | 42.0 | 32.7 | 42.0 | | NNW | Lightly overcast—light wind. |
| ♂ 22 | 30.027 | 42.3 | 30.145 | 43.2 | 38 | 38.7 | 39.9 | 36.3 | 39.9 | 0.017 | SSW | Overcast.—P.M. Light rain and wind. |
| ☿ 23 | 30.403 | 41.3 | 30.354 | 43.8 | 37 | 38.5 | 41.6 | 35.3 | 43.8 | | NNW | Overcast. |
| ♃ 24 | 30.139 | 43.7 | 30.079 | 44.8 | 44 | 44.6 | 46.6 | 37.2 | 46.6 | | W | Foggy. |
| ♀ 25 | 29.824 | 46.7 | 29.780 | 48.3 | 44 | 44.6 | 45.8 | 42.0 | 45.8 | | W | A.M. Foggy. P.M. Fine—light clouds and wind. |
| ○ ♄ 26 | 29.190 | 46.6 | 29.227 | 49.3 | 45 | 45.3 | 46.8 | 39.7 | 51.3 | 0.039 | WSW | Fine—light clouds. Showery. |
| ☉ 27 | 29.272 | 45.2 | 29.220 | 48.6 | 41 | 41.7 | 50.3 | 36.7 | 51.3 | 0.125 | S | A.M. Rain. P.M. Fine and clear—brisk wind. |
| ☽ 28 | 29.537 | 44.7 | 29.576 | 47.4 | 39 | 39.8 | 45.7 | 36.3 | 46.5 | 0.181 | WSW | Fine and clear—light haze. |
| | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Sum | | |
| | 29.837 | 45.1 | 29.842 | 47.3 | 41.4 | 42.8 | 46.0 | 38.5 | 47.2 | 1.445 | | |

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.802 } { 3 P.M. 29.800 }

## OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) = 95 feet.

The external Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

# METEOROLOGICAL JOURNAL FOR MARCH, 1831.

| 1831. March. | 9 o'clock, A.M. Barom. | 9 o'clock, A.M. Attach. Therm. | 3 o'clock, P.M. Barom. | 3 o'clock, P.M. Attach. Therm. | Dew Point at 9 A.M. in degrees of Fahr. | External Thermometer. Fahrenheit. 9 A.M. | Fahrenheit. 3 P.M. | Self-registering. Lowest. | Self-registering. Highest. | Rain, in inches. Read off at 9 A.M. | Direction of the Wind at 9 A.M. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ♂ 1 | 29.844 | 40.1 | 29.897 | 45.4 | 38 | 38.8 | 46.5 | 34.3 | 46.5 | | W | Fine and cloudless—light haze. |
| ☿ 2 | 29.751 | 44.7 | 29.691 | 48.6 | 47 | 47.3 | 53.5 | 37.8 | 53.7 | 0.036 | WSW | Overcast. Light fog A.M. |
| ♃ 3 | 29.622 | 49.7 | 29.641 | 52.5 | 51 | 51.4 | 54.0 | 46.6 | 54.3 | 0.075 | SW | A.M. Foggy—light rain. P.M. Overcast. |
| ♀ 4 | 29.764 | 52.7 | 29.897 | 55.0 | 49 | 49.8 | 52.8 | 49.8 | 53.7 | 0.222 | NW | A.M. Foggy. P.M. Fine—light clouds and wind. |
| ♄ 5 | 29.830 | 52.8 | 29.641 | 54.2 | 49 | 49.4 | 51.3 | 47.3 | 51.3 | | SSW | Overcast—light wind. |
| ☉ 6 | 29.154 | 53.2 | 29.161 | 54.3 | 51 | 51.3 | 51.3 | 48.6 | 51.8 | | S var. | Lightly cloudy—light unsteady wind. |
| ☽ 7 | 29.684 | 51.3 | 29.776 | 53.4 | 47 | 47.7 | 50.5 | 43.8 | 50.7 | | NW | Fine and clear—light clouds. |
| ♂ 8 | 29.753 | 48.6 | 29.594 | 52.5 | 45 | 45.3 | 50.8 | 38.3 | 51.4 | 0.022 | ESE | Cloudy—light wind and fog. |
| ☿ 9 | 29.595 | 49.4 | 29.612 | 52.6 | 44 | 44.4 | 49.8 | 40.5 | 50.3 | | SW | A.M. Fine and cloudless. P.M. Hail and rain with light wind. |
| ♃ 10 | 29.926 | 46.8 | 29.938 | 50.7 | 39 | 39.7 | 50.6 | 35.3 | 50.6 | 0.061 | W | Fine and cloudless. |
| ♀ 11 | 29.714 | 51.7 | 29.597 | 53.4 | 49 | 49.6 | 51.6 | 38.7 | 51.6 | 0.028 | SSE | Overcast—light rain and wind. |
| ♄ 12 | 29.937 | 49.0 | 29.756 | 53.0 | 43 | 43.4 | 49.8 | 37.6 | 51.3 | | WSW | A.M. Fine. P.M. Light rain and wind. |
| ☉ 13 | 29.660 | 49.7 | 29.607 | 51.0 | 45 | 46.8 | 46.7 | 40.8 | 48.7 | | WSW | Overcast. Heavy showers, with light brisk wind. |
| ● ☽ 14 | 29.690 | 47.9 | 29.681 | 50.3 | 41 | 44.3 | 45.5 | 39.3 | 49.7 | 0.086 | WSW | Fine and clear—light clouds. Hail and rain P.M. |
| ♂ 15 | 29.778 | 47.3 | 29.659 | 50.0 | 43 | 43.7 | 49.7 | 40.3 | 52.3 | 0.042 | SW | Light rain. |
| ☿ 16 | 29.648 | 50.8 | 29.741 | 53.3 | 51 | 52.7 | 55.7 | 43.2 | 56.3 | 0.014 | W | Cloudy—light wind. |
| ♃ 17 | 29.851 | 53.3 | 29.893 | 55.8 | 53 | 54.3 | 57.6 | 52.3 | 57.7 | | SSW var. | Fine—cloudy. |
| ♀ 18 | 30.184 | 52.3 | 30.196 | 55.4 | 52 | 46.3 | 50.8 | 40.3 | 41.0 | | W NW | Fine and clear—light clouds. |
| ♄ 19 | 30.217 | 49.7 | 30.172 | 53.5 | 42 | 43.8 | 51.0 | 37.7 | 51.5 | | WSW | Fine—lightly cloudy. |
| ☉ 20 | 30.112 | 49.6 | 30.072 | 52.7 | 43 | 44.7 | 52.7 | 41.8 | 52.7 | | W | Lightly cloudy—light wind and fog. |
| ☽ 21 | 30.092 | 51.4 | 30.097 | 54.2 | 49 | 49.5 | 54.4 | 44.3 | 54.4 | | N | Overcast—hazy. |
| ♂ 22 | 30.262 | 50.7 | 30.283 | 53.0 | 45 | 46.7 | 48.3 | 43.7 | 48.3 | 0.063 | ESE | Fine—light clouds. |
| ☿ 23 | 30.369 | 46.9 | 30.297 | 47.8 | 34 | 42.4 | 42.3 | 37.7 | 44.6 | | E | Fine—lightly cloudy—light brisk wind. |
| ♃ 24 | 29.958 | 41.3 | 29.811 | 43.2 | 35 | 36.1 | 38.4 | 31.7 | 40.3 | | N | Overcast—light brisk wind.—Snow A.M. |
| ♀ 25 | 29.653 | 44.3 | 29.570 | 47.7 | 39 | 41.7 | 44.6 | 34.3 | 44.7 | | E | Lightly cloudy—light wind. |
| ♄ 26 | 29.345 | 45.3 | 29.429 | 49.8 | 43 | 43.7 | 54.3 | 40.3 | 54.7 | 0.061 | E | A.M. Showers, with fog and light wind. P.M. Fine. |
| ☉ 27 | 29.906 | 51.4 | 29.941 | 53.7 | 51 | 51.7 | 53.3 | 42.6 | 59.4 | 0.153 | S | Fine and clear—light clouds. |
| ○ ☽ 28 | 30.056 | 51.3 | 30.053 | 55.3 | 47 | 47.7 | 56.7 | 42.3 | 59.4 | | N | Fine and clear—light clouds.—Hazy A.M. |
| ♂ 29 | 30.165 | 47.7 | 30.168 | 48.0 | 42 | 42.7 | 44.7 | 39.7 | 45.0 | | N | Overcast—light wind. |
| ☿ 30 | 30.281 | 45.3 | 30.277 | 49.6 | 35 | 41.6 | 48.2 | 38.5 | 49.7 | | N | Overcast. Hazy A.M. |
| ♃ 31 | 30.401 | 44.6 | 30.398 | 48.6 | 38 | 42.4 | 48.0 | 37.3 | 48.3 | | N | Fine and clear. Clouds and haze A.M. |
| | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Sum | | |
| | 29.877 | 48.7 | 29.856 | 51.6 | 44.5 | 45.8 | 50.2 | 40.9 | 50.8 | 0.883 | | |

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.834 } { 3 P.M. 29.801 }

## OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about)..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

# METEOROLOGICAL JOURNAL FOR APRIL, 1831.

| 1831. April. | 9 o'clock, A.M. | | 3 o'clock, P.M. | | Dew Point at 9 A.M. in degrees of Fahr. | External Thermometer. | | | | Rain, in inches. Read off at 9 A.M. | Direction of the Wind at 9 A.M. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Barom. | Attach. Therm. | Barom. | Attach. Therm. | | Fahrenheit. | | Self-registering. | | | | |
| | | | | | | 9 A.M. | 3 P.M. | Lowest. | Highest. | | | |
| ♀ 1 | 30.448 | 45.9 | 30.335 | 48.3 | 37 | 44.0 | 45.7 | 37.5 | 46.3 | | N | Cloudy—light wind. Rain at night. |
| ♄ 2 | 30.071 | 44.7 | 30.025 | 47.2 | 37 | 41.3 | 44.7 | 36.4 | 45.7 | 0.085 | SE | Lightly cloudy. |
| ☉ 3 | 29.963 | 46.3 | 29.853 | 50.2 | 43 | 44.3 | 51.3 | 40.3 | 51.3 | | N | Lightly cloudy—light wind. |
| ☽ 4 | 29.702 | 45.3 | 29.590 | 48.5 | 40 | 41.3 | 48.4 | 36.7 | 48.5 | | N | Overcast—hazy. |
| ♂ 5 | 29.503 | 45.3 | 29.468 | 49.2 | 40 | 42.7 | 49.5 | 35.3 | 51.3 | | SSW | Cloudy—hazy. |
| ☿ 6 | 29.504 | 49.7 | 29.468 | 52.1 | 43 | 47.3 | 54.5 | 39.7 | 55.3 | | SW | Fine—lightly cloudy. |
| ♃ 7 | 29.375 | 49.7 | 29.261 | 54.5 | 48 | 49.7 | 59.0 | 43.3 | 59.7 | | ESE | Overcast—light wind. |
| ♀ 8 | 29.284 | 52.3 | 29.306 | 52.8 | 49 | 50.7 | 48.6 | 48.3 | 52.7 | | S | Overcast—light rain. |
| ♄ 9 | 29.503 | 52.9 | 29.534 | 56.6 | 51 | 53.5 | 56.6 | 45.7 | 58.3 | 0.058 | SSE | Lightly cloudy. Light rain at night. |
| ☉ 10 | 29.610 | 56.7 | 29.698 | 58.6 | 52 | 54.7 | 57.4 | 49.4 | 58.6 | 0.325 | S | Clear—lightly cloudy—light wind. |
| ☽ 11 | 29.981 | 54.7 | 29.976 | 57.2 | 50 | 52.3 | 58.7 | 45.7 | 58.7 | | N | Lightly overcast. |
| ♂ 12 | 29.925 | 54.6 | 29.830 | 58.3 | 53 | 53.6 | 60.8 | 48.3 | 61.8 | | NE | Fine—lightly cloudy. At 9½ P.M. Thunder and lightning, with rain. |
| ☿ 13 | 29.820 | 57.7 | 29.794 | 61.6 | 57 | 57.6 | 63.8 | 48.4 | 64.8 | 0.153 | N | Lightly cloudy. |
| ♃ 14 | 29.969 | 58.3 | 29.948 | 59.7 | 43 | 50.7 | 56.8 | 48.6 | 56.8 | | N | Fine and clear—light clouds and wind. |
| ♀ 15 | 30.015 | 57.3 | 29.985 | 58.3 | 46 | 52.7 | 55.8 | 43.7 | 57.1 | | W | Lightly cloudy. |
| ♄ 16 | 30.064 | 58.2 | 30.006 | 59.9 | 50 | 56.7 | 57.4 | 57.2 | 59.3 | | N | Lightly cloudy. |
| ☉ 17 | 29.989 | 55.4 | 29.994 | 57.7 | 49 | 49.7 | 52.3 | 49.2 | 55.3 | | N | Cloudy—clear. Light rain A.M. |
| ☽ 18 | 30.044 | 52.7 | 29.977 | 55.9 | 39 | 46.6 | 56.2 | 38.3 | 56.3 | | N | Fine and clear—light clouds. |
| ♂ 19 | 29.953 | 49.7 | 29.877 | 54.7 | 45 | 45.7 | 57.8 | 39.9 | 57.8 | | N | Light wind.—A.M.: overcast. P.M. Fine and clear. |
| ☿ 20 | 29.815 | 53.3 | 29.720 | 55.9 | 41 | 48.6 | 56.2 | 40.4 | 56.7 | | N | Fine—lightly cloudy. |
| ♃ 21 | 29.533 | 51.4 | 29.453 | 54.3 | 47 | 47.7 | 53.4 | 43.7 | 53.8 | | NNE | Overcast. |
| ♀ 22 | 29.429 | 54.3 | 29.429 | 58.9 | 53 | 54.4 | 60.2 | 45.8 | 61.7 | | E | Fine—lightly cloudy. |
| ♄ 23 | 29.508 | 53.8 | 29.513 | 60.3 | 52 | 52.3 | 60.8 | 47.8 | 65.2 | | NNW | A.M. Overcast. Noon, hail and rain. P.M. Fine. |
| ☉ 24 | 29.825 | 56.7 | 29.873 | 57.3 | 50 | 52.6 | 54.7 | 49.7 | 54.7 | 0.019 | N | Overcast. |
| ☽ 25 | 30.005 | 56.7 | 29.960 | 60.3 | 51 | 54.7 | 62.8 | 50.3 | 63.7 | | NNW | Cloudy—light wind. |
| ♂ 26 | 29.817 | 59.3 | 29.753 | 62.3 | 49 | 57.6 | 61.8 | 48.7 | 63.3 | | E | Lightly cloudy. |
| ☿ 27 | 29.646 | 56.6 | 29.546 | 61.7 | 49 | 52.5 | 58.7 | 49.7 | 60.7 | | SE | Lowering—light wind and fog. Rain at 10 A.M. |
| ♃ 28 | 29.347 | 56.3 | 29.316 | 58.4 | 51 | 51.7 | 54.0 | 50.3 | 55 6 | 0.036 | S | Overcast and foggy—deposition of moisture. |
| ♀ 29 | 29.203 | 56.3 | 29.216 | 61.3 | 50 | 50.5 | 61.0 | 45.5 | 61.3 | 0.367 | E | A.M. Foggy. P.M. Fine—cloudy—brisk wind—showery. |
| ♄ 30 | 29.243 | 60.4 | 29.195 | 62.4 | 51 | 58.0 | 61.0 | 49.7 | 63.7 | 0.033 | S | Fair—cloudy. |
| | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Sum | | |
| | 29.736 | 53.4 | 29.697 | 56.5 | 47.2 | 50.5 | 56.0 | 45.1 | 57.2 | 1.074 | | |

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.676 | 3 P.M. 29.631 }

## OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... ..... =83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

# METEOROLOGICAL JOURNAL FOR MAY, 1831.

| 1831. May. | 9 o'clock, A.M. Barom. | 9 o'clock, A.M. Attach. Therm. | 3 o'clock, P.M. Barom. | 3 o'clock, P.M. Attach. Therm. | Dew Point at 9 A.M. in degrees of Fahr. | External Thermometer. Fahrenheit. 9 A.M. | 3 P.M. | Self-registering. Lowest. | Highest. | Rain, in inches. Read off at 9 A.M. | Direction of the Wind at 9 A.M. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ☉ 1 | 29.462 | 58.7 | 29.418 | 60.3 | 53 | 57.3 | 59.0 | 46.7 | 63.3 | 0.019 | SSW | Clear—cloudy—showery. |
| ☽ 2 | 29.557 | 59.7 | 29.583 | 60.6 | 51 | 57.8 | 56.5 | 46.7 | 60.2 | 0.269 | E | Cloudy and foggy—showery. |
| ♂ 3 | 29.638 | 56.5 | 29.652 | 61.7 | 54 | 56.3 | 62.2 | 49.0 | 63.4 | 0.178 | SE | A.M. Cloudy—showers. P.M. Fine and clear—light brisk wind. |
| ☿ 4 | 29.599 | 61.3 | 29.527 | 61.9 | 53 | 60.0 | 54.7 | 49.4 | 62.3 | 0.017 | SSW | A.M. Fine and clear. P.M. Overcast. |
| ♃ 5 | 29.530 | 55.6 | 29.502 | 56.7 | 42 | 51.6 | 50.3 | 43.7 | 55.7 | | WSW | Fine—lightly cloudy. |
| ♀ 6 | 29.783 | 58.3 | 29.813 | 55.7 | | 48.5 | 48.7 | 40.3 | 55.7 | 0.153 | NW var. | Cloudy.—Fine and clear A.M. |
| ♄ 7 | 30.026 | 50.4 | 30.063 | 54.3 | | 44.7 | 50.3 | 33.8 | 54.5 | | N | Fine—light clouds. |
| ☉ 8 | 30.264 | 55.3 | 30.276 | 55.2 | 38 | 50.4 | 51.3 | 36.7 | 52.3 | | ENE | Cloudy.—Fine A.M. |
| ☽ 9 | 30.337 | 55.3 | 30.258 | 55.3 | 37 | 51.2 | 55.5 | 38.3 | 55.7 | | NNE | Fine and clear.—Cloudless P.M. |
| ♂ 10 | 30.215 | 51.3 | 30.162 | 55.7 | 47 | 47.5 | 54.4 | 43.5 | 54.6 | | ESE | Cloudy.—Light rain A.M. |
| ● ☿ 11 | 30.201 | 56.7 | 30.197 | 56.5 | 45 | 53.8 | 57.5 | 38.8 | 58.5 | | NE | Fine and clear—brisk wind. Cloudless P.M. |
| ♃ 12 | 30.125 | 55.3 | 30.038 | 57.2 | 47 | 53.7 | 60.0 | 39.5 | 61.7 | | NNE | Cloudless—hazy. |
| ♀ 13 | 29.965 | 61.4 | 29.942 | 61.9 | 51 | 57.7 | 61.7 | 45.5 | 63.8 | | NNE | A.M. Hazy. P.M. Fine and clear—light clouds. |
| ♄ 14 | 30.088 | 56.8 | 30.077 | 58.7 | 32 | 50.0 | 52.3 | 41.7 | 55.4 | | NNE | Lightly cloudy. |
| ☉ 15 | 30.058 | 58.5 | 30.010 | 59.9 | 35 | 53.5 | 62.6 | 39.4 | 63.7 | | ENE | Fine.—A.M. Light haze. P.M. Clear—light clouds. |
| ☽ 16 | 30.132 | 59.7 | 30.141 | 61.6 | 43 | 56.7 | 65.7 | 44.4 | 67.3 | | WSW | Fine—lightly overcast. |
| ♂ 17 | 30.215 | 63.4 | 30.151 | 63.7 | 48 | 63.3 | 64.3 | 48.3 | 67.2 | | ESE | Clear and cloudless. |
| ☿ 18 | 30.036 | 66.5 | 29.964 | 65.6 | 49 | 63.7 | 66.4 | 50.5 | 66.5 | | ENE | Clear and cloudless—light wind. |
| ♃ 19 | 29.811 | 67.7 | 29.704 | 65.3 | 45 | 61.7 | 63.8 | 52.3 | 66.2 | | E | Light wind. A.M. Cloudless. P.M. Cloudy. |
| ♀ 20 | 29.832 | 68.2 | 29.830 | 66.9 | 55 | 65.7 | 66.7 | 56.3 | 71.5 | | S | A.M. Fine and clear. A.M. Cloudy—showery. |
| ♄ 21 | 29.912 | 61.9 | 29.943 | 68.0 | 57 | 57.4 | 66.7 | 53.6 | 67.6 | 0.086 | N | A.M. Overcast. P.M. Fine and clear—light clouds. |
| ☉ 22 | 30.001 | 62.3 | 29.991 | 64.5 | 53 | 57.5 | 61.7 | 53.0 | 66.6 | | NNW | Lightly overcast. |
| ☽ 23 | 29.892 | 62.3 | 29.842 | 67.7 | 58 | 58.7 | 70.7 | 52.3 | 73.2 | | NNW | A.M. Overcast, P.M. Nearly cloudless |
| ♂ 24 | 29.840 | 64.3 | 29.817 | 69.7 | 60 | 60.5 | 69.7 | 55.7 | 72.6 | | N | A.M. Overcast. P.M. Nearly cloudless |
| ☿ 25 | 29.860 | 65.9 | 29.846 | 70.8 | 60 | 64.7 | 71.7 | 54.6 | 72.5 | | N | Fine and clear—light clouds and wind. |
| ○ ♃ 26 | 29.849 | 57.7 | 29.804 | 66.8 | 55 | 57.6 | 67.3 | 51.5 | 69.2 | | E | A.M. Overcast. P.M. Clear and cloudless. |
| ♀ 27 | 29.792 | 60.7 | 29.790 | 65.3 | 52 | 52.7 | 63.7 | 50.6 | 64.8 | | NNE | Lightly cloudy—light wind. |
| ♄ 28 | 29.929 | 58.7 | 29.971 | 64.7 | 47 | 57.7 | 63.6 | 50.7 | 65.3 | | N | Lightly cloudy. |
| ☉ 29 | 29.975 | 59.5 | 29.932 | 61.7 | 54 | 54.7 | 55.5 | 52.8 | 55.6 | | E | Rain. |
| ☽ 30 | 29.910 | 59.7 | 29.934 | 63.7 | 50 | 55.3 | 62.3 | 49.7 | 63.7 | 0.333 | NNE | A.M. Cloudy. P.M. Fine—light clouds |
| ♂ 31 | 29.997 | 63.7 | 29.942 | 63.4 | 50 | 61.3 | 62.3 | 51.3 | 63.6 | | E | Fine and clear—light clouds. |
| | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Sum | | |
| | 29.930 | 59.7 | 29.907 | 62.0 | 49.0 | 56.2 | 62.2 | 47.1 | 63.0 | 1.055 | | |

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.851 | 3 P.M. 29.821 }

## OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

# METEOROLOGICAL JOURNAL FOR JUNE, 1831.

| 1831. June. | 9 o'clock, A.M. Barom. | 9 o'clock, A.M. Attach. Therm. | 3 o'clock, P.M. Barom. | 3 o'clock, P.M. Attach. Therm. | Dew Point at 9 A.M. in degrees of Fahr. | External Thermometer. Fahrenheit. 9 A.M. | Fahrenheit. 3 P.M. | Self-registering. Lowest. | Self-registering. Highest. | Rain, in inches. Read off at 9 A.M. | Direction of the Wind at 9 A.M. | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ☿ 1 | 29.937 | 66.8 | 29.981 | 66.4 | 52 | 61.7 | 64.7 | 49.6 | 67.3 | | NNE | Fine and clear—light clouds and wind. |
| ♃ 2 | 30.153 | 67.7 | 30.113 | 66.4 | 44 | 62.7 | 70.0 | 48.6 | 70.6 | | N | Fine—light clouds and wind. |
| ♀ 3 | 30.188 | 66.7 | 30.157 | 67.7 | 52 | 58.7 | 70.3 | 49.9 | 70.3 | | E | Cloudless—hazy. |
| ♄ 4 | 30.196 | 64.4 | 30.188 | 68.5 | 47 | 60.7 | 68.7 | 48.7 | 69.5 | | NNE | Fine—light clouds and wind. |
| ☉ 5 | 30.136 | 69.7 | 30.081 | 71.7 | 52 | 67.7 | 73.4 | 55.3 | 74.8 | | W | Fine—clouds and haze—light wind. Rain P.M. |
| ☽ 6 | 30.081 | 65.3 | 30.078 | 65.6 | 43 | 57.4 | 59.4 | 51.3 | 60.3 | 0.025 | N | Fine and clear—cloudy—light brisk wind. |
| ♂ 7 | 30.074 | 60.4 | 30.007 | 63.3 | 42 | 54.3 | 58.4 | 49.9 | 61.3 | | NNW | Cloudy—light wind. |
| ☿ 8 | 29.847 | 65.7 | 29.826 | 66.9 | 52 | 61.0 | 67.3 | 49.5 | 67.6 | | NNW | Fine—cloudy—light wind. |
| ♃ 9 | 29.878 | 68.8 | 29.843 | 67.7 | 51 | 65.7 | 70.4 | 51.4 | 72.7 | | WSW | Fine—light clouds. Light rain P.M. |
| ● ♀ 10 | 29.752 | 73.4 | 29.771 | 69.1 | 52 | 65.6 | 67.4 | 57.5 | 70.7 | 0.047 | W | Fine—cloudy. |
| ♄ 11 | 29.634 | 69.4 | 29.734 | 68.7 | 53 | 65.3 | 70.4 | 57.7 | 71.7 | 0.011 | WSW | Clear—cloudy—light showers and unsteady wind. |
| ☉ 12 | 29.798 | 68.7 | 29.877 | 70.3 | 58 | 67.6 | 70.7 | 60.3 | 72.3 | | WSW | Fair—lightly cloudy. |
| ☽ 13 | 29.902 | 68.3 | 29.985 | 68.7 | 58 | 64.7 | 68.5 | 59.3 | 71.7 | 0.028 | W | Cloudy—light wind. Showers through the day. |
| ♂ 14 | 30.173 | 69.4 | 30.137 | 69.6 | 58 | 66.7 | 72.6 | 57.7 | 73.7 | 0.289 | W | Clear and cloudless. |
| ☿ 15 | 29.914 | 76.2 | 29.871 | 70.3 | 56 | 69.8 | 66.5 | 56.7 | 74.7 | | SSW | A.M. Very clear. P.M. Light rain. |
| ♃ 16 | 29.856 | 70.2 | 29.830 | 69.4 | 52 | 67.3 | 69.7 | 57.3 | 73.5 | 0.036 | WSW | Fine and clear—cloudy. |
| ♀ 17 | 29.948 | 67.7 | 29.987 | 66.7 | 55 | 65.6 | 67.0 | 55.7 | 69.3 | | WSW | Clear—cloudy—showery. |
| ♄ 18 | 30.019 | 64.7 | 30.026 | 67.9 | 63 | 63.6 | 67.0 | 56.7 | 69.6 | 0.055 | SSE var. | A.M. Lowering—light wind. P.M. Clear—Cloudy. |
| ☉ 19 | 29.970 | 69.7 | 30.023 | 70.0 | 56 | 67.6 | 71.0 | 60.7 | 73.3 | | SSW | A.M. Cloudy—light wind. P.M. Cloudless. |
| ☽ 20 | 30.205 | 68.7 | 30.205 | 69.7 | 50 | 67.8 | 71.5 | 53.4 | 73.4 | | W | A.M. Very clear. P.M. Fine—cloudy. |
| ♂ 21 | 30.226 | 73.4 | 30.175 | 71.5 | 53 | 69.7 | 75.3 | 55.6 | 75.7 | | S | Fine and clear—nearly cloudless. |
| ☿ 22 | 30.230 | 70.5 | 30.220 | 71.6 | 53 | 69.5 | 73.4 | 56.7 | 73.8 | | WSW | Fine—light haze—cloudy. |
| ♃ 23 | 30.232 | 70.4 | 30.180 | 71.3 | 57 | 67.7 | 71.0 | 58.5 | 72.7 | | NNE | Fine—hazy. |
| ♀ 24 | 29.941 | 65.8 | 29.854 | 67.7 | 55 | 61.3 | 66.7 | 57.8 | 68.7 | 0.103 | WSW | A.M. Rain. P.M. Fine and clear—light wind. |
| ○ ♄ 25 | 29.791 | 65.7 | 29.763 | 64.9 | 51 | 63.6 | 61.3 | 55.7 | 66.0 | 0.011 | W | Cloudy. Light showers P.M. |
| ☉ 26 | 29.636 | 64.3 | 29.658 | 66.3 | 55 | 58.6 | 62.5 | 53.4 | 67.6 | 0.150 | NNW | Fine and clear—light unsteady wind. Light rain, morning and evening. |
| ☽ 27 | 29.928 | 69.6 | 29.850 | 66.3 | 55 | 66.7 | 64.6 | 51.7 | 69.4 | 0.128 | WSW | A.M. Fine—light clouds. P.M. Rain. |
| ♂ 28 | 29.927 | 62.8 | 29.984 | 65.4 | 57 | 58.7 | 60.7 | 55.5 | 62.3 | 0.225 | NE | Cloudy—light brisk wind. |
| ☿ 29 | 30.034 | 62.6 | 30.004 | 66.2 | 53 | 57.3 | 63.4 | 54.4 | 65.6 | | N | Overcast. Light rain P.M. |
| ♃ 30 | 30.006 | 61.7 | 30.023 | 66.3 | 56 | 57.7 | 65.4 | 54.6 | 66.3 | 0.047 | NW | Overcast—light wind. |
| | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Mean | Sum | | |
| | 29.987 | 67.6 | 29.981 | 68.1 | 53.0 | 63.7 | 67.6 | 54.7 | 69.9 | 1.052 | | |

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## OBSERVANDA.

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.....above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

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